Interactions of computability and effective group theory.

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Plan

1. Definitions and a brief history.
2. Computable abelian groups.
3. Results.

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- The study of effective procedures in group theory pre-dates the formal definition of an algorithm (Max Dehn 1911).
- The systematic study of effective procedures in group theory was initiated by Higman, Rabin and Mal'cev in 1960's.
- Such studies can be generalized to other algebraic structures (Herrmann 1920's, van der Waerden 1930's, Rabin 1960's , Maltsev 1960's).

Definitions and a brief history

The main definition in this talk is:

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A computable group is not the same as a recursive group (Higman, 1960):

- A recursive group may have undecidable word problem;
- Higman mainly restricted himself to finitely generated groups.

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Definition

A group is

autostable or computably categorical

if there exists a computable isomorphism between any two isomorphic computable copies of this group.

Computable abelian groups

Problem

Describe abelian groups which are computable (relative to a given oracle).

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Limitwise monotonic approximations found applications:

- in computable linear orders (Harris),
- in computable models of $\aleph_1$-categorical theories (Khoussainov, Nies, Shore),
- in computable equivalence structures (Harizanova et al.),
- in a characterization of high c.e. degrees (Downey, Kach, Turetsky).

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The general case of arbitrary Ulm type is open.

Computable abelian groups

Problem (Khisamiev 1990's)

Describe computable groups of the from $\bigoplus_{p \in P} Q^{(p)}$, where P is a set of primes, and $Q^{(p)} = \{\frac{n}{p^k} : n \in Z \text{ and } k \in N\}$.

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Theorem (Downey, Goncharov, Knight et al. 2010)

The group G_P is computable if and only if P belongs to Σ^0_3 level of the Arithmetic Hierarchy.

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- There is no description of computable groups of the form $\bigoplus_i H_i$, where each $H_i \leq \mathbb{Q}$; such groups are called completely decomposable.

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- There is no description of computable groups of the form $\bigoplus_i H_i$, where each $H_i \leq \mathbb{Q}$; such groups are called completely decomposable.
- Even for the special case of Khisamiev's group the proof is not trivial.
- Completely decomposable groups have a nice algebraic structure theory developed by Baer (1930's).
- Countable torsion-free abelian groups (in general) have not been classified up to an isomorphism.

Completely decomposable groups

Thus, we restrict ourselves to:

Problem

Develop the theory of computable completely decomposable groups.

If all the elementary summands of $G = \bigoplus_{i \in \mathbb{N}} H_i$ are isomorphic, then G is said to be homogeneous.

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Theorem (Downey and M., 2011)

Every computable homogeneous completely decomposable group is

$$\Delta^0_3 \text{ categorical.}$$

Δ^0_3 = "computable in the second iteration of the Halting problem".

Completely decomposable groups

- For the proof of this theorem we introduce a new algebraic notion of S-independence:

Definition

For a set of primes $S \neq \emptyset$, we say that elements $b_1, \ldots, b_k$ of a group are S-independent if

$$p | \sum_{i \in \{1,\ldots,k\}} m_i b_i \text{ implies } \bigwedge_{i \in \{1,\ldots,k\}} p | m_i,$$

for all integers $m_1, \ldots, m_k$ and $p \in S$.

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- This notion is a natural generalization of the classical notion of p-independence to sets of primes.
- S-independence is crucial for the proof.

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Folklore (Follows from Nurtazin's results (1980's).)

A group of the form $G = \bigoplus_{i \in \mathbb{N}} H_i$ is never computably categorical.

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(That is, categorical relative to the Halting problem.)

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Remarkably, we can characterize when exactly this happens using

semi-low sets.

Completely decomposable groups

Definition (Soare 1982)

We say that a set S is semi-low if its weak jump

$$H_S = \{e : W_e \cap S \neq \emptyset\}$$

is computable in the Halting problem.

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is computable in the Halting problem.

Soare introduced semi-lowness to study the lattice $\mathcal{E}$ of c.e. sets under set-theoretical operations:

Theorem (Soare 1982)

If A is c.e. and coinfinite and $\overline{A}$ is semi-low then the principal filter $\{B : A \subseteq B \text{ and } B \in \mathcal{E}\}$ is isomorphic to $\mathcal{E}$.

Completely decomposable groups

Let $Q^{(P)}$ be the subgroup of $(Q, +)$ generated by

$$\{\frac{1}{p^m} : p \in P \text{ and } \mathrm{m} \in \omega\}.$$

Theorem (Downey and M.)

A homogeneous completely decomposable group $G = \bigoplus_{i \in \mathbb{N}} H$ is Δ_2^0-categorical if and only if $H \cong Q^{(P)}$, where P is a c.e. set of primes with a semi-low complement.

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- Different cases need different algebraic strategies.
- We introduce a new computably-theoretic concept of

  computable setting time

  which is crucial for the proof.

Completely decomposable groups

Definition (Computable setting time)

Let $\alpha = (h_i)_{i\in\omega}$ *be a sequence in* $(\omega \cup \{\infty\})^\omega$ *and suppose*

$$h_i = \sup_s h_{i,s},$$

for a computable double-sequence $(h_{i,s})_{i,s\in\omega}$.

We say that α *has a computable setting time if there is a (total) computable function* ψ *such that*

$$h_i = \begin{cases} h_{i,\psi(i)}, & \text{if } h_i \text{ is finite,} \\ \infty, & \text{otherwise,} \end{cases}$$

for every i.

Some further problems:

Problem

Does computable setting time have another applications in computable model theory?

Problem

Extend the results to other classes of completely decomposable abelian groups.

Problem

Study the effective content of S-independence.

Problem

Extend the results to free modules over computable rings.

Thank you!