The Converse of Deducibility: C.I. Lewis and the Origin of Modern Modal Logic

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Introductory Remarks

- Lewis famously held that strict implication represents deducibility.
- In later analyses of modal logic due to Dana Scott and (in a substructural setting) Kosta Došen modal logics are understood as representing their own rules of inference ("Modal Logic as Metalogic").
- This paper explores what Lewis meant by this and his programme is assessed.

Lewis founded modern modal logic, but Russell provoked him into it. For whereas there is much to be said for the material conditional as a version of 'if-then', there is nothing to be said for it as a version of 'implies'; and Russell called it implication, thus apparently leaving no place open for genuine deductive connections between sentences. Lewis moved to save the connections. But his way was not, as one could have wished, to sort out Russell's confusion of 'implies' with 'if-then'. Instead, preserving that confusion, he propounded a strict conditional and called it *implication.*

Quine, "Comments on Ruth Barcan Marcus "Modalities and Intensional Languages"" 1963

Early Lewisian criticism of material implication

Not only does the calculus of implication contain false theorems, but all its theorems are not proved. *For the theorems of the system are implied by the postulates in the sense of "implies" which the system uses. The postulates have not been shown to imply any of the theorems except in this arbitrary sense. Hence*, it has not been demonstrated *that the theorems can be inferred from the postulates, even if all the postulates are granted. The assumptions*, e. g., *of "Principia Mathematica," imply the theorems in the same sense that a false proposition implies anything, or the first half of any of the above theorems implies the last half.*

"Interesting Theorems in Symbolic Logic" 1913

- Lewis is claiming that in order to *prove a theorem*, one has to *show* that we can derive the theorem from the axioms.
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- But I think something more subtle is going on.
- He is claiming that all PM attempts to do is to show that either a theorem is actually true or that the axioms are actually false (not that this must be the case).
- This is not a real proof of the theorem.

Necessity and Deduction

> *Even if we take it to be the case that every truth implies every other, the process of* reasoning *about their relations must follow entirely different paths. It must proceed to ask* how *the one fact implies the other, and this inquiry turns upon possibilities which will seem to be wider than mere facts.*

"The Calculus of Strict Implication" 1914

What does a logic need to accomplish the task of showing that its theorems follow?

Lewis: a connective that captures the notion of deducibility.

Why isn't the turnstile good enough?

For Russell and Lewis in 1910-1920, '$\vdash A$' just means that A is being asserted as true. And they didn't really have the idea of hypotheses and entailment in the modern sense.

But, in 1932 ...

> *The answer is that, if pIq is a relation which sometimes holds when q is not deducible from p – and every truth-implication is such – then obviously, q cannot validly be inferred from p merely because pIq is true. But this is never done when pIq is made the basis of an inference in a truth-value system. The inference is made on the ground that pIq is a tautology. And the relation of p and q when pIq is a tautology is quite different from the relation when pIq is true but not tautological. Thus we may propound the conundrum: "When is a truth-implication not a truth-implication?" And the answer is: "When it is a tautology."*

Lewis and Langford, *Symbolic Logic*

Lewis's view seems to be this: when classical propositional calculus is understood as a logic properly so-called, we understand that only tautologies are being asserted. The turnstile is being taken to mean 'is a tautology' and so is, in effect, a modal operator.

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Logics, if they tell us what follows from what, are really modal logics.

"It appears that the relation of strict implication expresses precisely that relation which holds when valid deduction is possible. It fails to hold when valid deduction is not possible." (*Symbolic logic* 1932)

"Assume that $p \strictif q$ is to be true only when a conclusion q is logically inferable from a premise p." (Letter to Prior 1957)

S1

Definitions: $\alpha \vee \beta =_{df} \neg(\neg\alpha \wedge \neg\beta)$; $\alpha \prec \beta =_{df} \neg\Diamond(\alpha \wedge \neg\beta)$.

Axioms

1. $(p \wedge q) \prec (q \wedge p)$
2. $(p \wedge q) \prec p$
3. $p \prec (p \wedge p)$
4. $(p \wedge (q \wedge r)) \prec (q \wedge (p \wedge r))$
5. $((p \prec q) \wedge (q \prec r)) \prec (p \prec r)$
6. $p \prec \neg\neg p$
7. $p \prec \Diamond p$

Rules: *Strict Detachment*: $\vdash \alpha \prec \beta, \vdash \alpha \ \rightarrow \vdash \beta$; *Adjunction*; *Substitution for Strict Equivalence*; *Uniform Substitution* for propositional variables.

S2 and S3

S2 is S1 together with

$$\Diamond(p \wedge q) \prec \Diamond p$$

S3 is S2 together with

$$(p \prec q) \prec (\neg\Diamond q \prec \neg\Diamond p)$$

> *Those interested in the merely mathematical properties of such systems of symbolic logic tend to prefer more comprehensive and less 'strict' systems such as S5 and material implication. The interests of logical study would probably be best served by an exactly opposite tendency.*

Symbolic Logic

Failure of the Deduction Theorem in S1-S3

In 1946, Ruth Barcan shows that the deduction theorem fails S2:

$$A \strictif B \vdash \Diamond A \strictif \Diamond B$$

But

$$\nvdash (A \strictif B) \strictif (\Diamond A \strictif \Diamond B)$$

She obtains similar results for S1 and S3.

The key to these theorems is Barcan's application of the rules of the axiomatic system to hypotheses, rather than merely to the derivation of theorems from axioms.

Systems

A system is a set of states of affairs, T, such that

1. If $A \in T$ and $\vdash A \strictif B$ then $B \in T$;
2. If $A \in T$ and $B \in T$, then $A \wedge B \in T$.

Lewis Deducibility

A formula B is *Lewis deductible* from A if in every system in which A occurs, B occurs too.

Let's say that T' is a *first-order axiomatic basis* for T if T' together with the set of theorems of first-order logic and closed under modus ponens yields T.

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Theorem

If T' is a first-order axiomatic basis for T then the smallest system that contains T' is a conservative extension of T.

Lewis's Systems have to show too that their own theorems are deducible

But what does this mean?

This is a valid inference in *SL*:

1.	$p \prec (p \vee q)$	13.2
2.	$(p \prec p) \prec ((p \prec p) \vee p)$	1, *US*
3.	$p \prec p$	12.1
4.	$(p \prec p) \vee p$	2, 3, *MP*
5.	$(q \prec q) \vee q$	4, *US*

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But $\nvdash ((p \prec p) \vee p) \prec ((q \prec q) \vee q)$.

In S2 and S3 we do get

If B is deduced from axioms $A_1, ..., A_n$, then $\vdash (A_1 \wedge ... \wedge A_n) \dashv B$.

On the other hand ...

In S2 and S3, if A is any axiom and B is any theorem,

$$\vdash A \prec B$$

Maybe this isn't so bad?

In S2 and S3, if A and A' are axioms, then

$$\vdash A = A'$$

Final Remarks

- The failure of the deduction theorem doesn't really affect Lewis's programme.
- The analysis of deducibility itself does not seem to give a reason for adopting S1-S3 over S4 or S4 (or T ...).
- Lewis's programme of representing deducibility in his logic seems successful as far as it goes.
- It also is not very ambitious.
- But if he is right, all *real* logical systems are in fact modal.