The Difficult Point Conjecture for Graph Polynomials

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Graph polynomial project:
http://www.cs.technion.ac.il/~janos/RESEARCH/gp-homepage.html

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Graph polynomials

Graph polynomials

Graph polynomials are uniformly defined families of graph invariants which are (possibly) multivariate polynomials in some polynomial ring $\mathcal{R}$, usually $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. We find

- Graph polynomials as generating functions;
- Graph polynomials as counting certain types of colorings;
- Graph polynomials defined by recurrence relations;
- Graph polynomials as counting weighted homomorphisms (=partition functions).

A general study addresses the following:

- Representability of graph polynomials;
- How to compare graph polynomials;
- The distinguishing power of graph polynomials;
- Universality properties of graph polynomials;

Prominent (classical) graph polynomials

- The chromatic polynomial (G. Birkhoff, 1912)
- The Tutte polynomial and its colored versions
  (W.T. Tutte 1954, B. Bollobas and O. Riordan, 1999);
- The characteristic polynomial
  (T.H. Wei 1952, L.M. Lihtenbaum 1956, L. Collatz and U. Sinogowitz 1957)
- The various matching polynomials (O.J. Heilman and E.J. Lieb, 1972)
- Various clique and independent set polynomials (I. Gutman and F. Harary 1983)
- The Farrel polynomials (E.J. Farrell, 1979)
- The cover polynomials for digraphs (F.R.K. Chung and R.L. Graham, 1995)
- The interlace-polynomials
  (M. Las Vergnas, 1983, R. Arratia, B. Bollobás and G. Sorkin, 2000)
- The various knot polynomials (of signed graphs)
  (Alexander polynomial, Jones polynomial, HOMFLY-PT polynomial, etc)

Applications of classical graph polynomials

There are plenty of applications of these graph polynomials in

- Graph theory proper and **knot theory**;
- Chemistry and biology;
- Statistical mechanics (Potts and Ising models)
- **Social networks** and **finance mathematics**;
- Quantum physics and quantum computing

And what about the many other graph polynomials?

Outline of this talk

- Evaluations of graph polynomials
- Turing complexity vs BSS complexity
- The chromatic and the Tutte polynomial: A case study
- The Difficult Point Property (DPP)
- The class SOLEVAL as the BSS-analog for $\sharp\mathbf{P}$.
- The DPP Conjectures

Evaluations of graph polynomials

Evaluations of graph polynomials, I

Let $P(G;\bar{X})$ be a graph polynomial in the indeterminates $X_1,\ldots,X_n$.

Let $\mathcal{R}$ be a subfield of the complex numbers $\mathbb{C}$.

For $\bar{a} \in \mathcal{R}^n$, $P(-;\bar{a})$ is a graph invariant taking values in $\mathcal{R}$.

We could restrict the graphs to be from a class (graph property) $\mathcal{C}$ of graphs.

What is the complexity of computing $P(-;\bar{a})$ for graphs from $\mathcal{C}$?

- If for all graphs $G \in \mathcal{C}$ the value of $P(-;\bar{a})$ is a graph invariant taking values in $\mathbb{N}$, we can work in the Turing model of computation.

- Otherwise we identify the graph G with its adjacency matrix M_G, and we work in the Blum-Schub-Smale (BSS) model of computation.

Our goal

We want to discuss and extend the classical result of

F. Jaeger and D.L. Vertigan and D.J.A. Welsh

on the complexity of evaluations of the Tutte polynomial. They show:

- either evaluation at a point $(a, b) \in \mathbb{C}^2$ is polynomial time computable in the Turing model, and a and b are integers,
- or some $\sharp\mathbf{P}$-complete problem is reducible to the evaluation at $(a, b) \in \mathbb{C}^2$.
- To stay in the Turing model of computation, they assume that (a, b) is in some finite dimensional extension of the field $\mathbb{Q}$.

The proof of the second part is a hybrid statement:
The reduction is more naturally placed in the BSS model of computation,
However, $\sharp\mathbf{P}$-completeness has no suitable counterpart in the BSS model.

It seems to us **more natural** to work **entirely**
in the BSS model of computation.

Evaluations of graph polynomials, II

- A graph invariant or graph parameter is a function $f : \bigcup_n \{0,1\}^{n \times n} \to \mathcal{R}$ which is invariant under permutations of columns and rows of the input adjacency matrix.

- A graph transformation is a function $T : \bigcup_n \{0,1\}^{n \times n} \to \bigcup_n \{0,1\}^{n \times n}$ which is invariant under permutations of columns and rows of the input adjacency matrix.

- The BSS-P-time computable functions over $\mathcal{R}$, $P_{\mathcal{R}}$, are the functions $f : \{0,1\}^{n \times n} \to \mathcal{R}$ BSS-computable in time $O(n^c)$ for some fixed $c \in \mathbb{N}$.

- Let f_1, f_2 be graph invariants. f_1 is BSS-P-time reducible to f_2, $f_1 \leq_P f_2$ if there are BSS-P-time computable functions T and F such that

  (i) T is a graph transformation ;

  (ii) For all graphs G with adjacency matrix M_G we have
  $$f_1(M_G) = F(f_2(T(M_G)))$$

- two graph invaraints f_1, f_2 are BSS-P-time equivalent, $f_1 \sim_{BSS-P} f_2$, if $f_1 \leq_{BSS_P} f_2$ and $f_2 \leq_{BSS_P} f_1$.

Evaluations of graph polynomials, III: Degrees and Cones

What are difficult graph parameters in the BSS-model?

Let g, g' be a graph parameters computable in exponential time in the BSS-model, i.e., $g, g' \in EXP_{BSS}$.

BSS-Degrees We denote by $[g]_{BSS}$ and $[g]_T$ the equivalence class (BSS-degree) of all graph parameters $g' \in EXP_{BSS}$ under the equivalence relation $\sim_{BSS-P}$.

BSS-Cones We denote by $< g >_{BSS}$ the class (BSS-cone) $\{g' \in EXP_{BSS} : g \leq_{BSS-P} g'\}$.

NP-completeness There are BSS-NP-complete problems, and instead of specifing them, we consider NP to be a degree (which may vary with the choice of the Ring $\mathcal{R}$).

NP-hardness The cone of an NP-complete problem forms the NP-hard problems.

Decision problems, functions and graph parameters

- The BSS model deals traditionally with decision problems where the input is an $\mathcal{R}$-vector.

- A function f maps $\mathcal{R}$-vectors into $\mathcal{R}$.
  $f(\bar{X}) = a$ becomes a decision problem.

- There is no well developed theory of degrees and cones of functions in the BSS model.

- In the study of graph polynomials decision problems and functions have as input $(0,1)$-matrices and the decision problems and functions have to be graph invariants.

Evaluations of graph polynomials, IV

We work in BSS model over $\mathcal{R}$.

We define

$$\text{EASY}_{BSS}(P, \mathcal{C}) = \{\bar{a} \in \mathcal{R}^n : P(-; \bar{a}) \text{ is BSS-P-time computable } \}$$

and

$$\text{HARD}_{BSS}(P, \mathcal{C}) = \{\bar{a} \in \mathcal{R}^n : P(-; \bar{a}) \text{ is BSS-NP-hard } \}$$

We use $\text{EASY}_{BSS}(P)$ and $\text{HARD}_{BSS}(P)$ if $\mathcal{C}$ is the class of all finite graphs.

How can we describe EASY$(P, \mathcal{C})$ and HARD$(P, \mathcal{C})$?

Turing Complexity vs BSS complexity

Problems with hybrid complexity, I

Let f_1, f_2 be two graph parameters taking values in $\mathbb{N}$

as a subset of the ring $\mathcal{R}$.

We have two kind of reductions:

- T-P-time Turing reductions (via oracles) in the Turing model.
  $f_1 \leq_{T-P} f_2$ iff f_1 can be computed in T-P-Time using f_2 as an oracle.

- BSS-P-time reductions over the ring $\mathcal{R}$.
  $f_1 \leq_{BSS-P} f_2$ iff f_1 can be computed in BSS-P-Time using f_2 as an oracle.

- In the Turing model there is a natural class of problems $\sharp\mathbf{P}$ for counting, problems which contains many evaluation of graph polynomials.
  However, $\sharp\mathbf{P}$ is **NOT CLOSED** under T-P-reductions.

- In the BSS model no corresponding class seems to accomodate graph polynomials.

Problems with hybrid complexity, II

- We shall propose a new candidate, the class $\text{SOLEVAL}_{\mathcal{R}}$ of evaluations of $\mathbf{SOL}$-polynomials, the graph polynomials definable in Second Order Logic as described by T. Kotek, JAM, and B. Zilber (2008, 2011).
- The main problem with hybrid complexity is the apparent incompatibility of the two notions of polynomial reductions, $f_1 \leq_{T-P} f_2$ and $f_1 \leq_{BSS-P} f_2$ even in the case where f_1 and f_2 are both in $\sharp\mathbf{P}$.
- The number of 3-colorings of a graph, $\sharp$3COL, and the number of acyclic orientations $\sharp$ACYCLOR are T-P-equivalent, and $\sharp\mathbf{P}$-complete in the Turing model.
- In the BSS model we have $\sharp$3COL $\leq_{BSS-P}$ $\sharp$ACYCLOR, but it is open whether $\sharp$ACYCLOR $\leq_{BSS-P}$ $\sharp$3COL holds.

Case study:

The chromatic polynomial and the Tutte polynomial

The (vertex) chromatic polynomial

Let $G = (V(G), E(G))$ be a graph, and $\lambda \in \mathbb{N}$.

A **λ-vertex-coloring** is a map

$$c : V(G) \to [\lambda]$$

such that $(u, v) \in E(G)$ implies that $c(u) \neq c(v)$.

We define $\chi(G, \lambda)$ to be the number of λ-vertex-colorings

Theorem: (G. Birkhoff, 1912)

$\chi(G, \lambda)$ is a **polynomial in $\mathbb{Z}[\lambda]$**.

Proof:

(i) $\chi(E_n) = \lambda^n$ where E_n consists of n isolated vertices.

(ii) For any edge $e = E(G)$ we have $\chi(G - e, \lambda) = \chi(G, \lambda) + \chi(G/e, \lambda)$.

Interpretation of $\chi(G,\lambda)$ for $\lambda \notin \mathbb{N}$

What's the point in considering $\lambda \notin \mathbb{N}$?

R. Stanley, 1973 For simple graphs G, $|\,\chi(G,-1)\,|$ counts the number of **acyclic orientations** of G.

R. Stanley, 1973 There are also combinatorial interpretations of $\chi(G,-m)$ for each $m \in \mathbb{N}$, which are more complicated to state.

Open: What about $\chi(G,\lambda)$ for each $m \in \mathbb{R} - \mathbb{Z}$?

Open: What about $\chi(G,\lambda)$ for each $m \in \mathbb{C} - \mathbb{R}$?

The complexity of the chromatic polynomial, I

Theorem:

- $\chi(G, 0)$, $\chi(G, 1)$ and $\chi(G, 2)$ are P-time computable (Folklore)
- $\chi(G, 3)$ is $\sharp\mathbf{P}$-complete (Valiant 1979).
- $\chi(G, -1)$ is $\sharp\mathbf{P}$-complete (Linial 1986).

Question:

What is the complexity of computing $\chi(G, \lambda)$ for

$\lambda = \lambda_0 \in \mathbb{Q}$

or even for

$\lambda = \lambda_0 \in \mathbb{C}$?

The complexity of the chromatic polynomial, II

Let $G_1 \bowtie G_2$ denote the join of two graphs.

We observe that

$$\chi(G \bowtie K_n, \lambda) = (\lambda)^{\underline{n}} \cdot \chi(G, \lambda - n) \tag{$\star$}$$

Hence we get

(i) $\chi(G \bowtie K_1, 4) = 4 \cdot \chi(G, 3)$

(ii) $\chi(G \bowtie K_n, 3 + n) = (n + 3)^{\underline{n}} \cdot \chi(G, 3)$
hence for $n \in \mathbb{N}$ with $n \geq 3$ it is $\sharp\mathbf{P}$-complete.

This works in the Turing model of computation

for λ in some Turing-computable field extending $\mathbb{Q}$.

The complexity of the chromatic polynomial, III

If we have have an oracle for some $q \in \mathbb{Q} - \mathbb{N}$ which allows us to compute $\chi(G, q)$ we can compute $\chi(G, q')$ for any $q' \in \mathbb{Q}$ as follows:

Algorithm $A(q, q', | V(G) |)$**:**

(i) Given G the degree of $\chi(G, q)$ is at most $n =| V(G) |$.

(ii) Use the oracle and $(\star)$ to compute $n + 1$ values of $\chi(G, \lambda)$.

(iii) Using Lagrange interpolation we can compute $\chi(G, q')$ in polynomial time.

We note that this algorithm is purely algebraic and works for all graphs G, $q \in (F) - \mathbb{N}$ and $q' \in F$ for any field F extending $\mathbb{Q}$.

Hence we get that for all $q_1, q_2 \in \mathbb{C} - \mathbb{N}$ the graph parameters are polynomially reducible to each other.

Furthermore, for $3 \leq i \leq j \in \mathbb{N}$, $\chi(G, i)$ is reducible to $\chi(G, j)$.

This works in the BSS-model of computation.

The complexity of the chromatic polynomial, IV

We summarize the situation for the chromatic polynomial as follows:

(i) $\text{EASY}_{BSS}(\chi) = \{0, 1, 2\}$ and $\text{HARD}_{BSS}(\chi) = \mathbb{C} - \{0, 1, 2\}$.

(ii) $\text{HARD}_{BSS}(\chi)$ can be split into two sets:

(ii.a) $\text{HARD}_{\sharp\mathbf{P}}(\chi)$: the graph parameters which are counting functions in $\sharp\mathbf{P}$ in the sense of Valiant, with $\chi(-, 3) \leq_P \chi(-, j)$ for $j \in \mathbb{N}$ and $3 \leq j$.
All graph parameters in $\text{HARD}_{\sharp\mathbf{P}}(\chi)$ are $\sharp\mathbf{P}$-complete in the Turing model.

(ii.b) $\text{HARD}_{BSS-NP}(\chi)$: the graph parameters which are not counting functions.
In the BSS model they are all polynomially reducible to each other,
and all graph parameters in $\text{HARD}_{\sharp\mathbf{P}}(\chi)$ are P-reducible to each of the graph parameters in $\text{HARD}_{BSS}(\chi)$.

(ii.c) In the BSS-model the graph parameter $\chi(-, 3)$ is P-reducible to all the parameters in $\text{HARD}_{BSS}(\chi)$.

(ii.d) Inside $\text{HARD}_{BSS}(\chi)$ we have:

$$\chi(-, 3) \leq_{BSS_P} \chi(-, 4) \leq_{BSS_P} \ldots \chi(-, j) \ldots \leq_{BSS-P} \chi(-, a) \sim_{BSS_P} \chi(-, -1)$$

with $j \in \mathbb{N} - \{0, 1, 2\}$ and $a \in \mathbb{C} - \mathbb{N}$.

The complexity of the chromatic polynomial, V

We have a Dichotomy Theorem for the evaluations of $\chi(-,\lambda)$:

(i) $\text{EASY}_{BSS}(\chi) = \{0, 1, 2\}$

Over $\mathbb{C}$ this is a quasi-algebraic set (a finite boolean combination of algebraic sets) of dimension 0.

(ii) All graph parameters in $\text{HARD}_{BSS}(\chi)$

are at least as difficult as $\chi(-, 3)$

(via BSS-P-reductions)

This is a quasi-algebraic set of dimension 1.

Evaluating the Tutte polynomial (Jaeger, Vertigan, Welsh)

The Tutte polynomial $T(G, X, Y)$ is a bivariate polynomial

and $\chi(G, \lambda) \leq_P T(G, 1 - \lambda, 0)$.

We have the following Dichotomy Theorem:

(i) $\text{EASY}_{BSS}(T) = \{(x, y) \in \mathbb{C}^2 : (x - 1)(y - 1) = 1\} \cup \text{Except}$, with
$\text{Except} = \{(0, 0), (1, 1), (-1, -1), (0, -1), (-1, 0), (i, -i), (-i, i), (j, j^2), (j^2, j)\}$
and $j = e^{\frac{2\pi i}{3}}$
Over $\mathbb{C}$ this is a quasi-algebraic set of dimension 1.

(ii) All graph parameters in $\text{HARD}_{BSS}(T)$ are at least as hard as $T(G, 1 - \lambda, 0)$.
This is a quasi-algebraic set of dimension 2.

The proof and its generalizations

- ($\star$) is replaced by two (or more) operations:

  stretching and **thickening**.

- Lagrange interpolation is done on a **grid**.

- There are considerable technical challenges in details of the proof for the **Tutte polynomial**.

- Allthough in all successfull generalizations to other cases, the same general outline of the proof is always similar, substantial challenges in the details have to be overcome.

Evaluations of graph polynomials, V: How hard is $\sharp$3COL $= \chi(-, 3)$?

- It is known that 3-colorability of graphs can be phrased as problem of solvability of quadratic equations and therefore is $\mathbf{NP}_{\mathbb{R}}$-hard and $\mathbf{NP}_{\mathbb{C}}$-hard in the BSS-model (Hillar and Lim, 2010).
- For $\mathbb{C}$, Malajovich and Meer (2001) proved an analogue of Ladner's Theorem for the BSS-model over $\mathbb{C}$:

  Assuming that $\mathbf{P}_{\mathbb{C}} \neq \mathbf{NP}_{\mathbb{C}}$ there are infinitely many different BSS-degrees between them.
- Although the problem $\chi(-, 3) \neq 0$? is $\mathbf{NP}_{\mathbb{C}}$-hard we do not know whether there is $a \in \mathbb{C} - \mathbb{N}$ for which computing $\chi(-, a)$ is really harder!
- In particular, we know that $\chi(a, 3) \leq_{BSS-P} \chi(-, -1)$,

  but we do not know whether

$$\chi(a, -1) \leq_{BSS-P} \chi(-, 3)$$

The Difficult Point Property (DPP)

Difficult Point Property, I

Given a graph polynomial $P(G, \bar{X})$ in n indeterminates $X_1, \ldots, X_n$ we are interested in the set $\text{HARD}_{BSS}(P)$.

(i) We say that P has the weak difficult point property (WDPP) if there is a quasi-algebraic subset $D \subset \mathbb{C}^n$ of co-dimension $\leq n-1$ which is contained in $\text{HARD}_{BSS}(P)$.

(ii) We say that P has the strong difficult point property (SDPP) if there is a quasi-algebraic subset $D \subset \mathbb{C}^n$ of co-dimension $\leq n-1$ such that $D = \text{HARD}_{BSS}(P)$ and $\mathbb{C} - D = \text{EASY}_{BSS}(P)$.

In both cases $\text{EASY}_{BSS}(P)$ is of dimension $\leq n-1$, and for almost all points $\overline{a} \in \mathbb{C}^n$ the evaluation of $P(-, \overline{a})$ is BSS-NP-hard.

$\chi(G; \lambda)$ and $T(G; X, Y)$ both have the SDPP.

Difficult Point Property, II

We compare WDPP and SDPP to Dichtomy Properties.

(i) We say that P has the dichotomy property (DiP) if
$\mathsf{HARD}_{BSS}(P) \cup \mathsf{EASY}_{BSS}(P) = \mathbb{C}^n$.
Clearly, if $\mathbf{P}_{\mathbb{C}} \neq \mathbf{NP}_{\mathbb{C}}$, $\mathsf{HARD}_{BSS}(P) \cap \mathsf{EASY}_{BSS}(P) = \emptyset$.

(ii) WDPP is not a dichtomy property, but SDPP a dichotomy property.

(iii) The two versions of DPP have a quantitative aspect:

$\mathsf{EASY}_{BSS}(P)$ **is small**.

The class $\mathrm{SOLEVAL}_{\mathcal{R}}$ as the BSS-analog for $\sharp\mathbf{P}$.

Uniform definability of subset expansions

of graph polynomials

in (Monadic) Second Order Logic SOL (MSOL)

After: T. Kotek and J.A. Makowsky and B. Zilber,
On Counting Generalized Colorings,
In: Model Theoretic Methods in Finite Combinatorics,
M. Grohe and J.A. Makowsky, eds.,
Contemporary Mathematics, vol. 558 (2011), pp. 207-242
American Mathematical Society,

Simple (M)SOL-graph polynomials

Let $ind(G, i)$ denote the number of independent sets of size i of a graph G.

The graph polynomial $ind(G, X) = \sum_i ind(G, i) \cdot X^i$, can be written also as

$$ind(G, X) = \sum_{I \subseteq V(G)} \prod_{v \in I} X$$

where I ranges over all independent sets of G.
To be an independent set is definable by a formula of Monadic Second Order Logic (MSOL) $\phi(I)$.

A **simple (M)SOL-definable graph polynomial** $p(G, X)$ is a polynomial of the form

$$p(G, X) = \sum_{A \subseteq V(G): \phi(A)} \prod_{v \in A} X$$

where A ranges over all subsets of $V(G)$ satisfying $\phi(A)$

and $\phi(A)$ is a (M)SOL-formula.

General (M)SOL-graph polynomials

For the general case

- One allows several indeterminates $X_1, \ldots, X_t$.
- One gives an inductive definition.
- One allows an ordering of the vertices.
- One requires the definition to be **invariant under the ordering**, i.e., different orderings still give the same polynomial.
- This also allows to define the modular counting quantifiers $C_{m,q}$ "there are, modulo q exactly m elements..."

The general case includes the chromatic polynomial and the Tutte polynomial and its variations, and virtually all graph polynomials from the literature.

1, 2, many SOL-definable graph polynomials

It is now easy to define many (also non-prominent) graph polynomials without further combinatorial motivation.

- Let $c_{i,j,k}(G)$ denote the number of triples of subsets of $A, B, C \subseteq E(G)$ such that $|A| = i, |B| = j, |C| = k$ and $\langle G, A, B, C \rangle \models \phi(A, B, C)$ where ϕ is any SOL-formula. Then
$$f(G; X, Y, Z) = \sum_{i,j,k} c_{i,j,k}(G) X^i Y^j Z^K$$
is an SOL-definable graph polynomial.

- The generalized chromatic polynomials introduced by Kotek, JAM and Zilber (2008) are all SOL-definable provided the coloring condition is also SOL-definable.

- The PhD theses of my students I. Averbouch and T. Kotek contain detailed studies of new graph polynomials with

combinatorial motivations.

The class SOLEVAL$_{\mathcal{R}}$

We now propose an analog to Valiant's counting class $\sharp\mathbf{P}$ for the BSS-model.

Assume $\mathbb{N} \subset \mathcal{R}$.

A function $f : \bigcup_n \{0,1\}^{n \times n} \to \mathcal{R}$ is in SOLEVAL$_{\mathcal{R}}$ if it is the evaluation of some SOL-definable graph polynomial.

Facts and questions:

Fact: If $f \in$ SOLEVAL$_{\mathcal{R}}$ then it is in EXPTIME$_{\mathcal{R}}$ and even in PSPACE$_{\mathcal{R}}$.

Fact: Every graph parameter $f \in \sharp\mathbf{P}$ is in SOLEVAL$_{\mathcal{R}}$.

Question: Is SOLEVAL$_{\mathcal{R}}$ contained in a single BSS-degree?
In particular, is $[\chi(-,-1)]_{BSS}$ its **maximal** degree?

Question: What is the BSS-degree structure of SOLEVAL$_{\mathcal{R}}$?

The DPP conjectures

The DPP conjectures

We have conjectured the following:

J.A. Makowsky,
From a Zoo to a Zoology: Towards a general theory of graph polynomials,
Theory of Computing Systems, vol. 43 (2008), pp. 542-562.

Let P be an SOL-definable graph polynomial in n indeterminates.

Assume that for some $\bar{a} \in \mathbb{C}^n$ evaluation of $P(-, \bar{a})$ is BSS-NP-hard over $\mathbb{C}$.

Weak DPP Conjecture: Then P has the WDPP.

Strong DPP Conjecture: Then P has the SDPP.

In the following we present more evidence for these conjectures.

Partition functions as graph polynomials

- Let $A \in \mathbb{C}^{n \times n}$ a symmetric and G be a graph. Let

  $$Z_A(G) = \sum_{\sigma: V(G) \to [n]} \prod_{(v,w) \in E(G)} A_{\sigma(v),\sigma(w)}$$

  Z_A is called a partition function.

- Let $\mathbf{X}$ be the matrix $(X_{i,j})_{i,j \leq n}$ of indeterminates.
  Then $Z_{\mathbf{X}}$ is a graph polynomial in n^2 indeterminates,
  Z_A is an evaluation of $Z_{\mathbf{X}}$, and $Z_{\mathbf{X}}$ is MSOL-definable.

Partition have the SDPP

- J. Cai, X. Chen and P. Lu (2010),
  building on A. Bulatov and M. Grohe (2005),
  proved a dichotomy theorem for $Z_{\mathbf{X}}$ where $\mathcal{R} = \mathbb{C}$.

- Analyzing their proofs reveals:
  $Z_{\mathbf{X}}$ satsifies the SDPP for $\mathcal{R} = \mathbb{C}$.

- There are various generalizations of this to Hermitian matrices,
  M. Thurley (2009),
  and beyond.

More SOL-definable graph polynomials with the DPP, I

SDPP: the cover polynomial $C(G, x, y)$ introduced
by Chung and Graham (1995)
by Bläser, Dell 2007, Bläser, Dell, Fouz 2011

SDPP: the bivariate matching polynomial for multigraphs,
by Averbouch and JAM, 2007

SDPP: the harmonious chromatic polynomial,
by Kotek and JAM, 2007

WDPP: the Bollobás-Riordan polynomial, generalizing the Tutte polynomial
and introduced by Bollobás and Riordan (1999),
by Bläser, Dell and JAM 2008, 2010.

WDPP: the interlace polynomial (aka Martin polynomial) introduced by Martin (1977) and independently by Arratia, Bollobás and Sorkin (2000),
by Bläser and Hoffmann, 2007, 2008

Generalized chromatic polynomials

Let $f: V(G) \to [k]$ be a coloring of the vertices of $G = (V(G), E(G))$.

(i) f is proper if $(uv) \in E(G)$ implies that $f(u) \neq f(v)$. In other words if for every $i \in [k]$ the counter-image $[f^{-1}(i)]$ induces an independent set.

(ii) f is convex if for every $i \in [k]$ the counter-image $[f^{-1}(i)]$ induces a connected graph.

(iii) f is t-improper if for every $i \in [k]$ the counter-image $[f^{-1}(i)]$ induces a graph of maximal degree t..

(iv) f is H-free if for every $i \in [k]$ the counter-image $[f^{-1}(i)]$ induces an H-free graph.

(v) f is acyclic if for every $i, j \in [k]$ the union $[f^{-1}(i)] \cup [f^{-1}(i)]$ induces an acyclic graph.

By Kotek, JAM, Zilber (2008), for all the above properties, counting the number of colorings is a polynomial in k.

More SOL-definable graph polynomials with the DPP, II

T. Kotek and JAM (2011) have shown

SDPP: The graph polynomial for convex colorings.

SDPP: The graph polynomial for t-improper colorings (for multigraphs).

SDPP: The bivariate chromatic polynomial introduced by Döhmen, Pönitz and Tittman in 2003.

WDPP: The graph polynomial for acyclic colorings.

C. Hoffmann's PhD thesis (written under M. Bläser, 2010) contains a general sufficient criterion which allows to establish the WDPP for a wide class of (mostly non-prominent) graph polynomials.

A good test problem: H-free colorings.

We look at the generalized chromatic polynomial $\chi_{H-free}(G;k)$, which, for $k \in \mathbb{N}$ counts the number of H-free colorings of G.

- For $H = K_2$, $\chi_{H-free}(G;k) = \chi(G;k)$, and we have the SDPP.
- For $H = K_3$, $\chi_{H-free}(G;k)$ counts the triangle free-colorings.
- From [ABCM98] it follows that $\chi_{H-free}(G;k)$ is #P-hard for every $k \geq 3$ and H of size at least 2.

  D. Achlioptas, J. Brown, D. Corneil, and M. Molloy. The existence of uniquely -G colourable graphs. *Discrete Mathematics*, 179(1-3):1–11, 1998.

- In [Achlioptas97] it is shown that computing $\chi_{H-free}(G;2)$ is NP-hard for every H of size at most 2.

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- **Characterize H for which $\chi_{H-free}(G;k)$ satisfies the SDPP (WDPP).**

Thank you for your attention !

Model Theoretic Methods in Finite Combinatorics

M. Grohe and J.A. Makowsky, eds.,
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Especially the papers

- Application of Logic to Combinatorial Sequences and Their Recurrence Relations
  E. Fischer, T. Kotek, and J. A. Makowsky
- On Counting Generalized Colorings
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