Σ_1 Induction and Proper d-r.e. Degrees

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The Twelfth Asian Logic Conference

17 December, 2011

Wellington

Preliminaries

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Definition 1

An expression ϕ is called a Σ_1 formula if it is in the form of

$$\exists x_1 \exists x_2 \ldots \exists x_n Q_1(x_{n+1} < y_1) \ldots Q_m(x_{n+m} < y_m) \\ \psi(x_1, \ldots, x_{n+m}; y_1, \ldots, y_m; a_1, \ldots, a_k)$$

where $Q_1, \ldots, Q_m$ are either $\forall$ or $\exists$, and ψ is conjunctions and disjunctions of equalities and inequalities of exponential polynomials.

Fragments of Peano Arithmetic

Peano Arithmetic (PA)

(1) $\forall x (S(x) \neq 0)$

(2) $\forall x \forall y (S(x) = S(y) \rightarrow x = y)$

(3) $\forall x (x + 0 = x)$

(4) $\forall x \forall y (x + S(y) = S(x + y))$

(5) $\forall x (x \times 0 = 0)$

(6) $\forall x (x \times S(y) = x \times y + x)$

(7) (Induction Schema) $(\forall y < x (\phi(y)) \rightarrow \phi(x)) \rightarrow \forall x (\phi(x))$

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(7) (Induction Schema) $(\forall y < x(\phi(y)) \rightarrow \phi(x)) \rightarrow \forall x(\phi(x))$

We use P^- to denote (1)-(6) plus that exponential functions are total.

Definition 2 (Σ_1 Induction ($\mathsf{I}\Sigma_1$))

P^- + Induction Schema restricted to Σ_1 formulae.

Bounding Schema

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For a formula $\phi(y, z)$*,*
$\forall y < x \exists z(\phi(y, z)) \rightarrow \exists b \forall y < x \exists z < b(\phi(y, z))$

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Definition 4 (Σ_1 Bounding (BΣ_1))

P^-+ *Bounding Schema restricted to* Σ_1 *formulae.*

Theorem 5 (Paris and Kirby)

$I\Sigma_1 \rightarrow B\Sigma_1 \not\rightarrow I\Sigma_1$

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Remark

In such a model, there is a Σ_1 definable proper initial segment closed under successor. It is a $\underline{\Sigma_1 \text{ cut}}$ I. There is a Δ_1 cofinal function f from the cut I to the whole model.

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- *A set X is regular iff for every finite set F, $X \cap F$ is finite.*

Definition 8

- $\underline{A \leq_T B}$ *iff there is an r.e. set* Φ *such that for any finite set* F*,*
  $F \subseteq A \Leftrightarrow \exists P \subseteq B \exists N \subseteq \bar{B}(\langle F, 1, P, N\rangle \in \Phi)$
  $F \subseteq \bar{A} \Leftrightarrow \exists P \subseteq B \exists N \subseteq \bar{B}(\langle F, 0, P, N\rangle \in \Phi)$

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  $F \subseteq \bar{A} \Leftrightarrow \exists P \subseteq B \exists N \subseteq \bar{B}(\langle F, 0, P, N\rangle \in \Phi)$
- *$A \equiv_T B$ iff $A \leq_T B$ and $B \leq_T A$*
- *The Turing degree of A: $\mathbf{a} = \{B : B \equiv_T A\}$.*

Definition 9

- *Turing degree* $\mathbf{a}$ *is r.e. if there is* $A \in \mathbf{a}$ *such that* A *is r.e.*

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- *Turing degree* $\mathbf{a}$ *is* <u>*d-r.e.*</u> *if there is* $D \in \mathbf{a}$ *such that* D *is d-r.e.*
- *Turing degree* $\mathbf{a}$ *is* <u>*proper d-r.e.*</u> *if it is d-r.e. but not r.e.*

Conclusion 10 (Mytilinaios)

If a theorem is provable using finite injury method, then it is provable in $I\Sigma_1$.

Turing Degrees in models of $\mathrm{B}\Sigma_1 + \neg\mathrm{I}\Sigma_1$

Theorem 11 (Slaman and Woodin)

In a model of $B\Sigma_1 + \neg I\Sigma_1$, its Σ_1 cut is of minimal degree.

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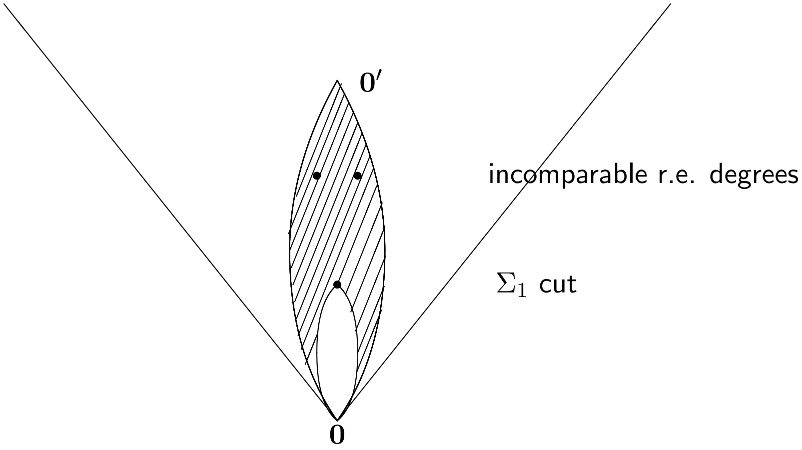
Theorem 13 (Chong and Mourad)

In a model of $B\Sigma_1 + \neg I\Sigma_1$, there are two incomparable r.e., that is, Friedberg-Muchnik Theorem holds.

Turing Degrees in models of $B\Sigma_1 + \neg I\Sigma_1$



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Corollary 15

$I\Sigma_1 \vdash$ *there exists a proper d-r.e. degree.*

Question

Does $\mathsf{B}\Sigma_1 + \neg\, \mathsf{I}\Sigma_1 \vdash$ there exists a proper d-r.e. degree?

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Ans: Yes.

Proper d-r.e. in $\mathrm{B}\Sigma_1 + \neg\ \mathrm{I}\Sigma_1$

Theorem 16

If $M \models B\Sigma_1 + \neg\ I\Sigma_1$, then there exists a d-r.e. set $D \not\leq_T \emptyset'$

Proper d-r.e. in $B\Sigma_1 + \neg I\Sigma_1$

Theorem 16

If $M \models B\Sigma_1 + \neg I\Sigma_1$, then there exists a d-r.e. set $D \not\leq_T \emptyset'$

Theorem 17

If $M \models B\Sigma_1 + \neg I\Sigma_1$, then there exists a bounded d-r.e. set $D \not\leq_T \emptyset'$

Theorem 18 (Chong and Mourad)

(Coding Lemma) If $X \subseteq I$ such that X and $I - X$ are Σ_1, then X is coded (there exists a finite set F with $F \cap I = X$).

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$A = \{x < 2^{a \times a} : \exists i \in I((x, i) \in \hat{H} \wedge (\forall j < i)(x, j) \notin \hat{H})\}$

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$B = \{x < 2^{a\times a} : \exists i < j \in I((x,i),(x,j) \in \hat{H})\}$

$D = A - B \subseteq [0, 2^{a\times a})$.

Claim: $D \not\leq_T \emptyset'$.

Question

$\mathsf{B}\Sigma_1 + \neg\, \mathsf{I}\Sigma_1 \vdash$ there exists a proper d-r.e. degree below $\mathbf{0}'$?

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Ans: No.

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Theorem 19

$B\Sigma_1 + \neg\ I\Sigma_1 \vdash$ there is no proper d-r.e. degree below $\mathbf{0}'$

Lemma 20

Suppose D *is d-r.e. and* $D \leq_T \emptyset'$.

(1) If D *is regular, then* $D \equiv_T \emptyset$ *or* $D \geq_T I$;

(2) if D *is not regular,* $D \geq_T I$.

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Suppose D is d-r.e. and $D \leq_T \emptyset'$.

(1) If D is regular, then $D \equiv_T \emptyset$ or $D \geq_T I$;

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Lemma 21

Suppose D is d-r.e. and $D \leq_T \emptyset'$. For every $m \in I$, there is some $b \in I$ such that

$$(\forall x \in [0, f(m)])[(x \in A[f(b)]) \vee (x \in B \to \exists N \subseteq \bar{\emptyset}'(\langle \{x\}, 0, N\rangle \in \Phi[f(b)]))]$$

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Ans. No.

Definition 22

Suppose $M \models B\Sigma_1 + \neg I\Sigma_1$ *and* $I = \omega$ *is a* Σ_1 *cut of* M.

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Theorem 23 (Slaman and Woodin)

There exists a saturated model.

Working in saturated models

Suppose M is saturated, and $D \leq_T \emptyset'$ via Φ.

Lemma 24

(1) If D is regular, then $D \equiv_T \emptyset$;

(2) if D is not regular, $D \geq_T I$.

honest, white liar and malicious liar

Pick any $i, m \in I$, let

$$D_{i,m} = \{x \in [0, f(m)] : \Phi^{\emptyset'}(x)[f(i)] = 1\}$$

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(1) there is no $j > i$ such that $D_{j,m}(x) \neq D_{i,m}(x)$.
Then we say x is i-<u>honest</u>.

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Then we say x is a i-white liar.

(3) there is some $j > i$ such that $D_{j,m}(x) \neq D_{i,m}(x)$, and $D_{i,m}(x) \neq D(x)$.
Then we say x is a i-malicious liar for stage i.

Lemma 25

If $D \leq_T \emptyset'$ then

for each $i \in I$ there is some $j \in I$ such that all i-white liars are found.

Tier Program II.

For each $m \in \omega$, we inductively construct a sequence of pairs $\{i_n^m\}_{n \in \omega} \subset \omega$ such that

(i) $i_0^m = 0$ and

(ii) for any n, i_{n+1}^m is the least stage j when all i_n^m-white liars in $[0, f(m)]$ are found.

As M is saturated, $\{\langle i_0^m, i_1^m, \ldots, i_{n(m)}^m \rangle\}_{m \in \omega}$ is coded.

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We may separate $[0, f(m)]$ into $n(m)+1$ parts:

$$K_i^m = \{x \in [0, f(m)] : g^m(x) = i\}$$

for each $i \leq n(m)$, where $g^m : [0, f(m)] \to \omega$,

$$x \mapsto \mu n(\forall k \in (i_n^m, i_{n+1}^m])(D_{i,m}(x) = D_{k,m}(x))$$

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$$x \mapsto \mu n(\forall k \in (i_n^m, i_{n+1}^m])(D_{i,m}(x) = D_{k,m}(x))$$

On K_i^m, enumerate $f(i)$-malicious liars, which determine $D \restriction K_i^m$.

Reference

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Thank you!