On Superstable Generic Structures

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Outline

1. *ab initio* **Generic Structures**
2. **Questions**
3. **Components**
4. **Amalgamation Classes**

Acknowledgement

This is a joint work with Koichiro Ikeda.

For a hyper-graph structure A, let

$$\delta(A) = \delta_\alpha(A) = |A| - \alpha e(A).$$

Here,
α is an real number such that $0 < \alpha \leq 1$,
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(A can be a relational structure with a relation symbol R, and $e(A) = |R(A)|$)
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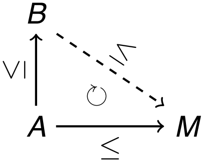
With this notation,

$$\mathbf{K}_\alpha = \{A : \text{finite} \mid A \geq \emptyset\}.$$

Suppose $\mathbf{K} \subseteq \mathbf{K}_\alpha$.
A countable hypergraph M is a generic structure of $\mathbf{K}$ if

- $A \subset_{\text{fin}} M \Rightarrow A \in \mathbf{K}$;
- $A \subseteq_{\text{fin}} M \Rightarrow$ there exists B such that $A \subseteq B \subseteq_{\text{fin}} M$ and $B \leq M$;
- for any $A, B \in \mathbf{K}$,



ab initio Generic Structures

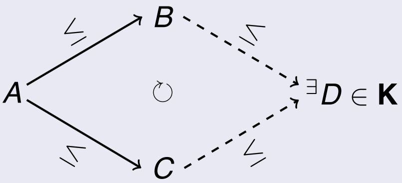
Fact

Suppose $\mathbf{K} \subseteq \mathbf{K}_\alpha$. If

1. $\emptyset \in \mathbf{K}$,
2. $A \subset B \in \mathbf{K} \Rightarrow A \in \mathbf{K}$,
3. and K has the AP, defined below,

then $\mathbf{K}$ has a generic structure.

K has the AP *iff* for any $A, B, C \in \mathbf{K}$,



- (Hrushovski) Stable, non-superstable, $\aleph_0$-categorical pseudoplane. (A counterexample to Lachlan's conjecture)
  $\delta(A) = |A| - \alpha e(A)$, α: irrational.
- (Hrushovski) CM-trivial strongly minimal set.
  $\delta(A) = |A| - e(A)$, $e(A) = |R(A)|$ with R a ternary relation.
- (Baldwin-Shelah) Sparse random graphs. (stable, non-superstable)
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Questions

Conjecture (Baldwin)

If an *ab initio* generic structure is superstable then it is ω-stable.

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Theorem (Ikeda)

For the predimension function $\delta(A) = |A| - e(A)$ there is a generic *graph* which is superstable but not ω-stable.

What about $\delta(A) = |A| - \alpha e(A)$ with $0 < \alpha < 1$?

Definition (minimal s-component)

Suppose $1 \leq s < 2$. A triple (E, a, b) with $a, b \in E$ is a minimal s-component if
for any substructure X of E such that $1 < |X| < |E|$,

1. $\delta(X) > 1$ if $a \notin X$ or $b \notin X$,
2. $\delta(X) > s$ if $a, b \in X$, and
3. $\delta(E) = s$.

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When $\delta(X) = |X| - e(X)$,  is a minimal 1-component.

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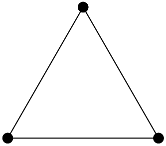
Proposition (K.)

If $\delta(X) = |X| - \alpha e(X)$ with a rational number α satisfying $0 < \alpha < 1$ then there is a minimal 1-component in $\mathbf{K}_\alpha$.
If G is a minimal 1-component then there is another minimal 1-component H such that $|H| > |G|$. Note that G cannot be embedded in H and H cannot be ebedded in G.

Minimal 1-Components

$\delta(A) = |A| - \frac{2}{3} e(A).$



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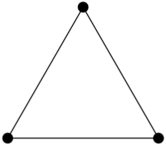

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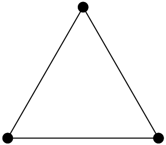

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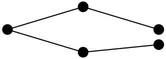

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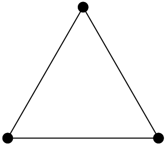

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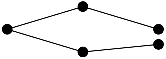

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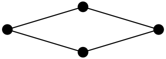

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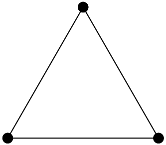

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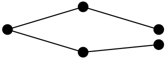

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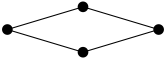

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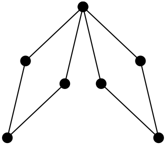

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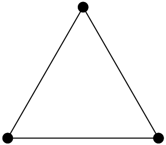

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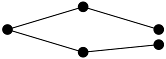

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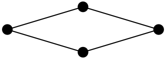

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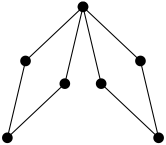

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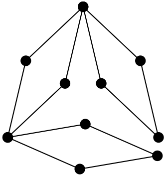

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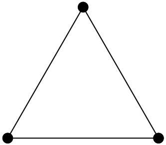

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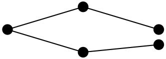

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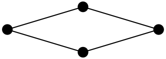

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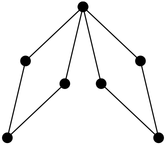

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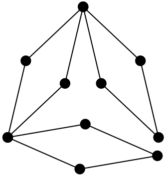

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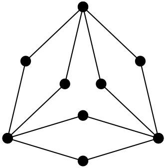

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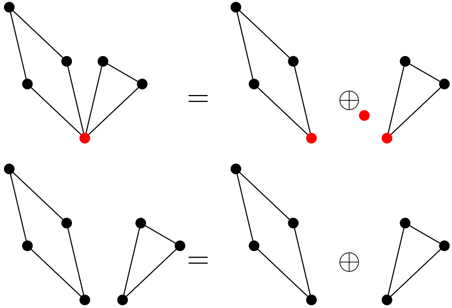
Suppose D: an R-structure, $A, B, C \subset D$.
$B \perp_A C$ *iff* $B \cap C = A$, and $R(D|(B \cup C)) = R(D|B) \cup R(D|C)$.
$D = B \oplus_A C$ *iff* $D = B \cup C$ and $B \perp_A C$.

Examples:



Let G, H be two minimal 1-components of different sizes.
Fix $a, b \in G$ and $c, d \in H$ with $a \neq b$ and $c \neq d$.
Suppose that there are hypergraphs A_1, A_2, ..., A_n and points $a_{i-1}, a_i \in A_i$ such that (A_i, a_{i-1}, a_i) is isomorphic to either (G, a, b) or (H, c, d) for each i, and

$$X = A_1 \oplus_{a_1} A_2 \oplus_{a_2} \cdots \oplus_{a_{n-1}} A_n.$$

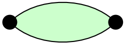
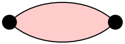
Such an X is called a GH-chain. If we can write

$$C = X/(a_0 = a_n)$$

then we call C a GH-cycle.
We can also define an *infinite GH-chain.*

GH-Chains and *GH*-Cycles

Minimal 1-components:

G :  and *H* : 

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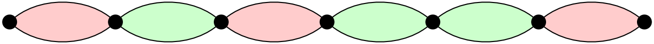

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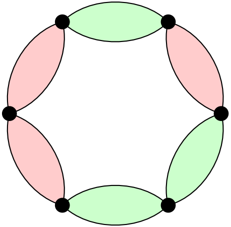

Suppose $0 < \alpha < 1$. Choose minimal 1-components (G, a, b), (H, c, d) such that they cannot be embedded mutually.
(If $\alpha = 1$ and $ar(R) \geq 3$ then we can choose such G and H also.)
Let
$\mathbf{E}$:= the class of GH-cycles,
$\mathbf{F} := \{D \mid \emptyset < D\}$,
i.e. $D \in \mathbf{F}$ iff $\delta(X) > 0$ for $X \subset D$ such that $X \neq \emptyset$,
$\mathbf{K} := \bigoplus(\mathbf{E} \cup \mathbf{F})$
(The class of free amalgams of finitely many structures from $\mathbf{E} \cup \mathbf{F}$).

Lemma (Ikeda-K.)

- $Y \in \mathbf{E} \Rightarrow \delta(Y) = 0$.
- $\emptyset \neq X \subsetneq Y \in \mathbf{E} \Rightarrow X \in \mathbf{F}$.
- $\mathbf{K}$ is an amalgamation class (hence, it has a generic structure).

The Generic Structure of **K**

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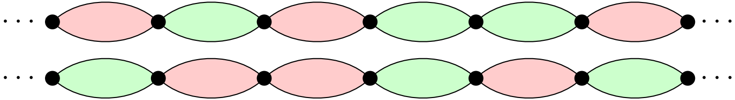
In a (countably) saturated $\tilde{M} \succ M$, there are all possible GH-chains of countable length which are also connected components of $\tilde{M}$.



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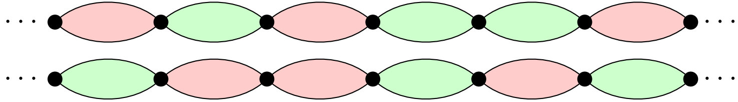

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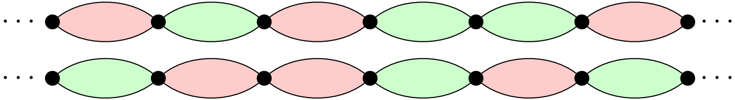

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Theorem (Ikeda-K.)

$Th(M)$ is not ω-stable.

M: The generic structure of **K**.
$\tilde{M}$: a saturated elementary extension of M.
For $a \in \tilde{M}$,

a is an i-element $:\iff a \in$ a GH-cycle which is also a connected component of $\tilde{M}$.

a is an s-element $:\iff a \in$ a GH-chain which is also a connected component of $\tilde{M}$.

a is an r-element $:\iff a$ is neither i- nor s-element.

For $S \subset \tilde{M}$, $U(S) :=$ the connected component of $\tilde{M}$ containing S.
$s(A) := \{a \in A \mid a \text{ is an s-element.}\}$.

Lemma

$A \le B \le \tilde{M}$, $A \le B' \le \tilde{M}$, B, B': finite, there is a partial isomorphism $f : \tilde{M} \to \tilde{M}$ such that $B \cup U(s(B)) \subset \mathrm{dom}(f)$, $f|A = \mathrm{id}_A$, $f(B) = B'$ and $f(U(s(B))) = U(s(B'))$
$\Rightarrow \quad \mathrm{tp}(B/A) = \mathrm{tp}(B'/A)$

1. For a finite set $A \subseteq \tilde{M}$, let $d(A) = \delta(\mathrm{cl}(A))$.
2. For finite sets $A, B \subseteq \tilde{M}$, let $d(B/A) = d(AB) - d(A)$.
3. For a finite set $B \in \tilde{M}$ and for any set $A \subseteq \tilde{M}$, let
   $d(B/A) = \inf\{d(B/A') \mid A' \text{ is a finite subset of } A\}$.
4. Suppose $A, B, C \subset \tilde{M}$. We write $B \mathop{\smile\!\!\!\!\!\mid}^{d}_{A} C$ if $B \perp_A C$ and $BC \leq \tilde{M}$.

Fact

Suppose $A, B, C \leq \tilde{M}$ and $B \cap C = A$. Then the following are equivalent:
- $d(B/C) = d(B/A)$.
- $B \mathop{\smile\!\!\!\!\!\mid}^{d}_{A} C$.

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For each cardinal $\lambda \geq 2^{\aleph_0}$, we show that $|S(N)| = \lambda$. Let $p \in S(N)$ and let $\bar{b}$ be a realization of p in $\tilde{M}$. We can choose a finite set $A \subset N$ such that $d(\bar{b}/N) = d(\bar{b}/A)$. Let $B = \mathrm{cl}(\bar{b}A)$. We can assume that $B \cap N = A$. By Fact, we have $B \downarrow^d_A N$.

Choose any $B' \models \mathrm{tp}(B/A)$ such that $B' \downarrow^d_A N$.

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Choose any $B' \models \mathrm{tp}(B/A)$ such that $B' \mathop{\smile\!\!\!\!\!\!|}^{d}_{A} N$.

Claim $\mathrm{tp}(B'/N) = \mathrm{tp}(B/N)$.

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Choose any $B' \models \mathrm{tp}(B/A)$ such that $B' \mathop{\smile\!\!\!\!\!\!\,|}^{d}_{A} N$.

Claim $\mathrm{tp}(B'/N) = \mathrm{tp}(B/N)$.

By Fact, we have $BN,\ B'N \leq \tilde{M}$ and $B \cong_N B'$. Since $\mathrm{tp}(B/A) = \mathrm{tp}(B'/A)$, we have $U(s(B)) \cong_A U(s(B'))$. Since N is a model, $U(s(B)) \perp_A N$ and $U(s(B')) \perp_A N$. Therefore, $U(s(B)) \cong_N U(s(B'))$. We have $\mathrm{tp}(B/N) = \mathrm{tp}(B'/N)$ by a lemma. We have the claim.
Now, it is easy to show that $|S(N)| = \lambda$.