Topological aspects of the degrees of difficulty of Π^0_1 classes

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We study interactions between *degrees of difficulty* and *effective topological properties* for $\mathbf{\Pi^0_1}$ classes.

Basic Fact

If we think of each set $S \subseteq \omega^\omega$ as the solution set of some mathematical problem, then any "*big*" set is "*easy*".
For example, any *dense* set is *easy*.

Question

What is the definition of *being easy*?

Definition

Let P, Q be problems (i.e., $P, Q \subseteq \omega^\omega$).

1. (Medvedev 1955) $P \leq Q$ if there is a (uniformly) computable map from Q to P, i.e.,
   $$P \leq Q \iff (\exists \Phi :\subseteq \omega^\omega \xrightarrow{\text{com}} \omega^\omega)(\forall \alpha \in Q)\ \Phi(\alpha) \in P.$$
2. (Muchnik 1963) $P \leq_w Q$ if there is a pointwise computable map from Q to P, i.e.,
   $$P \leq_w Q \iff (\forall \alpha \in Q)(\exists \Phi :\subseteq \omega^\omega \xrightarrow{\text{com}} \omega^\omega)\ \Phi(\alpha) \in P.$$

1. $P \leq Q$ (via Φ): If we have a solution α of Q, then we can calculate a solution of P by executing the algorithm Φ with an oracle α.
2. $P \leq_w Q$: If we have a solution α of Q, then there is a solution of P computable in α.

Fact

(Sorbi 1991, Lewis-Shore-Sorbi 2011) Every dense class $\boldsymbol{D} \subseteq \omega^\omega$ cups to no closed set $\boldsymbol{C} \subseteq \omega^\omega$.

$$(\forall \text{dense } \boldsymbol{d})(\forall \text{closed } \boldsymbol{c})(\forall \boldsymbol{z})\ \boldsymbol{d} \leq \boldsymbol{c} \vee \boldsymbol{z} \ \rightarrow\ \boldsymbol{d} \leq \boldsymbol{z}.$$

Remark

Every dense $\boldsymbol{\Pi}^0_1$ class is of degree $\mathbf{0}$.

Question

Is there a nontrivial "*nearly-dense*" property for $\boldsymbol{\Pi}^0_1$ classes which assures that these $\boldsymbol{\Pi}^0_1$ classes are *easy*?

Definition

1. Interval: $[\sigma] = \{\tau \in 2^{<\omega} : \tau \supseteq \sigma\}$.
2. P is dense iff P intersects all nonempty intervals.
3. P is *non-immune* if P intersects infinite c.e. many intervals.
   i.e., there is an infinite c.e. set $\mathcal{I}$ of intervals s.t. P intersects any $[\sigma] \in \mathcal{I}$.

Theorem (Cenzer-K.-Weber-Wu 2009)

Every non-immune Π^0_1 class is noncuppable.

$$\mathcal{P} \models (\forall \text{non-immune } a)(\forall z)\ z < 1 \ \rightarrow\ a \vee z < 1.$$

(Here, $\mathcal{P}$ denotes the distributive lattice consisting of Medvedev degrees of nonempty Π^0_1 classes of 2^ω)

Remark

1. T_P: a computable tree s.t. $[T_P] = P$.
2. T_P^{ext}: the set of all extendible nodes of T_P.
3. P is non-immune iff T_P^{ext} has no c.e. subtree.
4. (Def.) P is *non-tree-immune* if T_P^{ext} has no computable subtree.

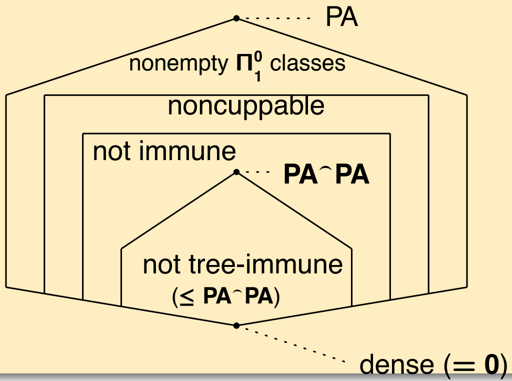
Definition

For $P, Q \subseteq X^\omega$, fix a symbol $. \notin X$.

$$Q^\frown P = [\{\sigma^\frown\langle . \rangle^\frown\tau : \sigma \in T_Q \;\&\; \tau \in T_P\}].$$

Theorem (Cenzer-K.-Weber-Wu 2009)

$P \leq \mathbf{PA}^\frown\mathbf{PA}$ iff P is not tree-immune.



Dense $\Longrightarrow$ Degree $\mathbf{0}$ $\Longrightarrow$ Not tree-immune $\Longleftrightarrow$ Below $\boldsymbol{PA}^\frown\boldsymbol{PA}$ $\Longrightarrow$ Not immune $\Longrightarrow$ Noncuppable

Theorem

$\mathbf{Q}^\frown\mathbf{P}$ does not cup to $\mathbf{P}$.

$$\mathcal{P} \models (\forall p, q, r)\ p \leq (q^\frown p) \vee r \ \rightarrow\ p \leq r.$$

Corollary

1. $\mathbf{P}^\frown\mathbf{P}$ is *strictly easier than* $\mathbf{P}$, in the sense of Medvedev.
2. $\mathbf{P}^\frown\mathbf{P}$ has *the same degree of difficulty* as $\mathbf{P}$, in the sense of Muchnik.

Corollary

Every nonzero $\mathbf{a} \in \mathcal{P}$ has the *strong anti-cupping property*:

$$\mathcal{P} \models (\forall a > 0)(\exists b)(\forall c)\ a \leq b \vee c \ \rightarrow\ a \leq c.$$

Definition (Gold 1965)

- Ψ is a learner if it is computable function from $\omega^{<\omega}$ to ω.
- A learner Ψ learns $\alpha \in \omega^\omega$ if $\Phi_{\lim_n \Psi(\alpha \upharpoonright n)} = \alpha$.

Definition (Relative Learnability)

- $\Gamma : \omega^\omega \to \omega^\omega$ is learnable if
  $(\exists \Psi)(\forall \alpha \in \mathbf{dom}(\Gamma))\ \Phi_{\lim_n \Psi(\alpha \upharpoonright n)}(\alpha) = \Gamma(\alpha)$.
  Here Ψ ranges over all learners.

Theorem

The following are equivalent:

1. A learnable map $f : Q \to P$ exists.
2. A computable map $f : Q \to \bigcup_n \underbrace{P^\frown P^\frown \dots {}^\frown P}_{n}$ exists.

Proposition

Let f be a partial map on ω^ω. The following are equivalent.

1. f is learnable.
2. f is a pointwise limit of a uniform sequence of computable maps on ω^ω with respect to the discrete topology on ω^ω.
3. f is piecewise computable, i.e., there is a Π^0_1 partition $\{P_i\}_{i\in\omega}$ of $\mathbf{dom}(f)$ such that $f \upharpoonright P_i$ is computable uniformly in $i \in \omega$.

Jayne-Rogers Theorem (1982)

Let f be a map on ω^ω. The following are equivalent.

1. f is (F_σ, F_σ)-measurable, i.e., the preimage $f^{-1}[A]$ is F_σ for any F_σ set $A \subseteq \omega^\omega$.
2. f is piecewise continuous, i.e., there is a closed partition $\{P_i\}_{i\in\omega}$ of $\mathbf{dom}(f)$ such that $f \upharpoonright P_i$ is continuous for any $i \in \omega$.

Conjecture (as an analogy of Jayne-Rogers Theorem)

A partial map on ω^ω is learnable if and only if it is effectively $(\boldsymbol{F_\sigma}, \boldsymbol{F_\sigma})$-measurable.

Problem

What is the definition of effective $(\boldsymbol{F_\sigma}, \boldsymbol{F_\sigma})$-measurability?

Definition

Let f be a partial map on ω^ω.

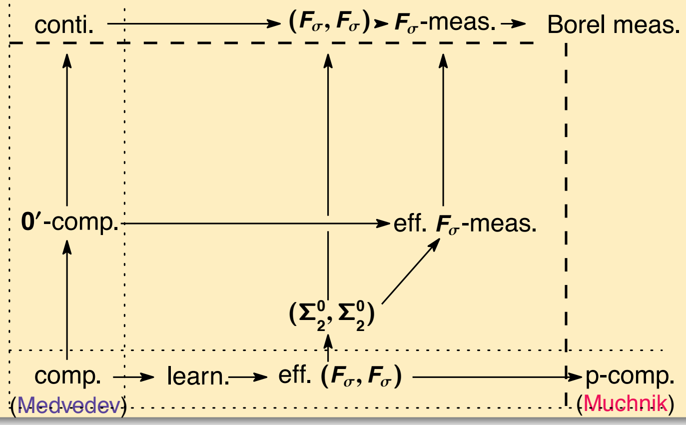
1. f is $(\boldsymbol{\Sigma}^0_2, \boldsymbol{\Sigma}^0_2)$-measurable if the preimage $f^{-1}[A]$ is $\boldsymbol{\Sigma}^0_2$ in $\mathrm{dom}(f)$, effectively in $\boldsymbol{\Sigma}^0_2$ sets $A \subseteq \omega^\omega$.
2. f is effective $(\boldsymbol{F_\sigma}, \boldsymbol{F_\sigma})$-measurable if the preimage $f^{-1}[A]$ is $\Sigma^0_2(\alpha)$ in $\mathrm{dom}(f)$, effectively in oracles $\alpha \in \omega^\omega$ and $\Sigma^0_2(\alpha)$ sets $A \subseteq \omega^\omega$.

Proposition

Learnable $\rightarrow$ eff. (F_σ, F_σ)-measurable $\Rightarrow$ $(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_2)$-measurable $\Rightarrow$ (F_σ, F_σ)-measurable.

Theorem

1. $(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_2)$-measurability $\nrightarrow$ pointwise computability.
   - Every $(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_2)$-meas. function f is uniformly low, i.e., $f(x)' \leq_T x'$ uniformly in x.
   - Every uniformly low function is $(\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_2)$-measurable.
2. Effectively (F_σ, F_σ)-measurability $\Rightarrow$ pointwise computability i.e., $f(x) \leq_T x$ for any $x \in \omega^\omega$.


conti.
(F_σ, F_σ)
F_σ-meas.
Borel meas.
$\mathbf{0}'$-comp.
eff. F_σ-meas.
(Σ^0_2, Σ^0_2)
comp.
learn.
eff. (F_σ, F_σ)
p-comp.
(Medvedev)
(Muchnik)

Theorem

1. There is no uniform computable function from $\mathbf{DNR_3}$ to $\mathbf{DNR_2}$. i.e., $\mathbf{DNR_3}$ is *easier* then $\mathbf{DNR_2}$, in the sense of Medvedev.
2. There is a pointwise computable function from $\mathbf{DNR_3}$ to $\mathbf{DNR_2}$. i.e., $\mathbf{DNR_3}$ has *the same degree of difficulty* as $\mathbf{DNR_2}$, in the sense of Muchnik.

Theorem

1. There is no learnable function from $\mathbf{DNR_3}$ to $\mathbf{DNR_2}$. Hence, there is no uniformly computable function from $\mathbf{DNR_3}$ to the *iterated concatenation of* $\mathbf{DNR_2}$ *along any well-founded tree*.
2. There is a uniformly computable function from $\mathbf{DNR_3}$ to the *iterated concatenation of* $\mathbf{DNR_2}$ *along the ill-founded tree* $T_{\mathbf{DNR_2}}$.

Remark

- "The *iterated concatenation of* $\boldsymbol{P}$ *along a well-founded tree*" (transfinite $^\frown$-iteration of $\boldsymbol{P}$) can be think of as a learning procedure to solve the problem $\boldsymbol{P}$ with an ordinal bound for mind changes.
- The following theorem suggests that "The *iterated concatenation of* $\boldsymbol{P}$ *along a ill-founded tree*" is *easier* then any transfinite $^\frown$-iteration of $\boldsymbol{P}$.

Definition

"The iterated concatenation of $\boldsymbol{P}$ along the ill-founded tree $\boldsymbol{T_Q}$" is called *the hyperconcatenation* of $\boldsymbol{Q}$ and $\boldsymbol{P}$.

Theorem

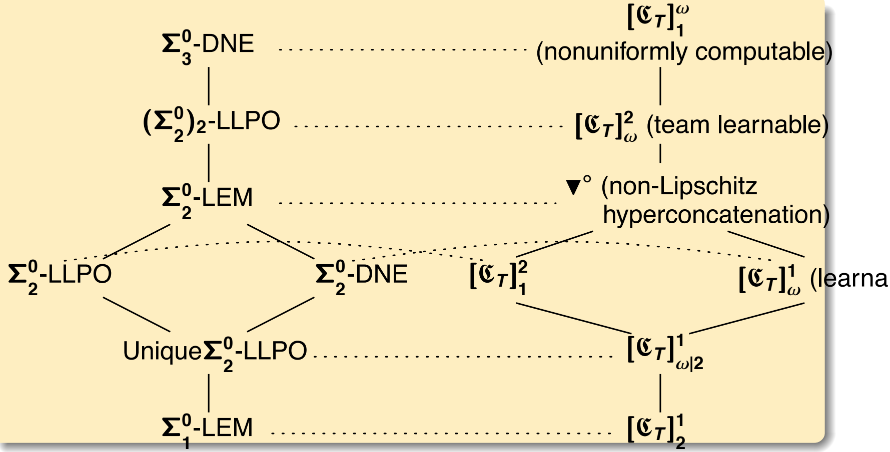
The hyperconcatenation of $\boldsymbol{Q}$ and $\boldsymbol{P}$ does not cup to $\boldsymbol{P}$, in the sense of learnability.

$$\mathcal{P}_l \models (\forall p, q, r)\ p \leq \mathrm{hyp}(q, p) \vee r \ \rightarrow \ p \leq r.$$

Theorem

$P, Q \subseteq \omega^\omega$.

1. The following are equivalent:
   1. There is a computable map from Q to the concatenation of P and P.
   2. There is a map $f : Q \to P$ which is Weihrauch reducible to the Σ^0_1-law of excluded middle.
2. The following are equivalent:
   1. There is a computable map from Q to the "*non-Lipschitz modification of*" the hyperconcatenation of P and P.
   2. There is a map $f : Q \to P$ which is Weihrauch reducible to the Σ^0_2-law of excluded middle.



- The law of excluded middle (LEM): $\varphi \lor \neg\varphi$
- The lessor limited principle of omniscience (LLPO): $\neg(\varphi \land \psi) \;\rightarrow\; \neg\varphi \lor \neg\psi$
- The double negation elimination (DNE): $\neg\neg\varphi \;\rightarrow\; \varphi$

Kojiro Higuchi, and Takayuki Kihara, *Inside the Muchnik degrees: discontinuity, learnability, and constructivism*, in preparation, 78 pages.

Thank you for your attention!