The relative consistency of an intuitionistic theory with functionals

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Intuitionistic logic is the most important non-classical logic since:

- philosophical justification and Brower's intuitionistic criticism of classical mathematics,
- many branches of mathematics have been developed within intuitionistic logic.

In intuitionistic logic a statement is considered true if it has an effective proof.

For example, if φ expresses an unsolved mathematical problem, then $(\varphi \vee \neg\varphi)$ is not considered true.

Next example belongs to van Dalen.

Theorem. There exist two irrational real numbers a and b, for which a^b is rational.

Classical proof

Consider the number $\sqrt{2}^{\sqrt{2}}$.

Case 1. If it is rational, then the theorem is proven.

Case 2. If it is irrational, take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$.

This proof does not clarify which case is true. The proof is not "effective" and is not acceptable from the intuitionistic point of view.

Intuitionistic proof

From a complex result of Gelfond it follows that $\sqrt{2}^{\sqrt{2}}$ is irrational.

So one can take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$.

The problem of establishing the consistency of classical set theory by means of intuitionistic theories:

Myhill, Friedman, Bernini and others.

Here:

Intuitionistic type theory LP, a modification of Bernini's theory:

- functionals of high types,
- lawless functionals,
- "creative subject".

A sub-theory I of the classical type theory:

- "almost predicative" axiom of separation,
- I is equiconsistent with LP,
- equiconsistency for each layer: I_s and LP_s are equiconsistent.

Here I_s and LP_s are the fragments of I and LP with types not higher than s.

Theory L. Language

Variables: $x, y, z,\ldots$ over natural numbers (variables of type 0);
for $n \geq 1$: $F^n, G^n, \ldots$ over n-functionals (functionals of type n);
$A^n, B^n, \ldots$ over constructive n-functionals;
$\mathcal{F}^n, \mathcal{G}^n, \mathcal{H}^n,\ldots$ over lawless n-functionals.

Constants: 0 of type 0, the constant K^n for each $n \geq 1$ (analogs of 0).

Functional symbols: N^n, Ap^n ($n \geq 1$) and a functional symbol for each primitive recursive function, including S for successor.

Predicate symbols: $=_n$ for each $n \geq 0$.

Terms and n-functionals are constructed from the constants and variables:

- Z is an n-functional $\Rightarrow$ $N^n(Z)$ is an n-functional (a successor of Z);
- Z is an 1-functional, t is a term $\Rightarrow$ $Ap^1(Z, t)$ is a term;
- Z is an $(n+1)$-functional, t is a term $\Rightarrow$ $Ap^n(Z, t)$ is an n-functional.

Denote $Ap^n(Z, t)$ as $Z(t)$; $Z(0)^m = \underbrace{Z(0)\ldots(0)}_{m \text{ times}}$.

Atomic formulas: $t =_0 \tau$, where t and τ are terms,
$Z =_n V$, where Z and V are n-functionals.

sort (φ) = the maximal type of parameters in formula φ (or 0 if φ has no parameters).

A 1-functional F^1 is interpreted as a function from ω to ω.

An (n+1)-functional F^{n+1} is interpreted as a function from ω to n-functionals.

A constructive functional is a function given by a rule.

A lawless functional is interpreted as usual in intuitionistic theories: at each point in time the researcher knows nothing about a lawless functional, except its initial segment.

Theory L has intuitionistic predicate logic with equality and these axioms:

1) $Sx \neq 0, \quad Sx = Sy \supset x = y$.

2) The defining relations for all primitive recursive functions.

3) Induction: $\varphi(0) \wedge \forall x\, (\varphi(x) \supset \varphi(Sx)) \supset \forall x \varphi(x)$.

4) Primitive recursive completeness of constructive functions:

$\exists A^1\, \forall x\, (A(x) = t)$, where t is any term containing only variables of type 0 and constructive 1-functionals.

5) $K^{n+1}(x) = K^n$, $\quad \neg(N^n(F^n) = K^n)$.

6) $N^{n+1}(F^{n+1})(x) = N^n(F(x))$, $\quad N^n(F^n) = N^n(G^n) \supset F = G$.

7) The axiom of choice with uniqueness:

$\forall x\, \exists!\, G^n\, \varphi(x,G) \supset \exists F^k\, \forall x\, \varphi(x, F(x)(0)^{k-n-1})$, where $k \geq \max(\mathrm{sort}(\varphi), n+1)$.

8) The axiom of choice for numbers:

$\forall x\, \exists y\, \varphi(x, y) \supset \exists F^n\, \forall x\, \varphi(x, F(x)(0)^{n-1})$, where $n \geq \max(\mathrm{sort}(\varphi), 1)$.

9) The axiom of existence of lawless functionals: $\exists \mathcal{F}^n\, (\forall y \leq x)\, (\mathcal{F}(y) = F^n(y))$

10) $\mathcal{F}^n = \mathcal{G}^n \;\vee\; \mathcal{F}^n \neq \mathcal{G}^n$.

11) The principle of open data:

$\varphi(\mathcal{F}^n) \supset \exists x\, \forall \mathcal{G}^n\, ((\forall y < x)(\mathcal{F}(y) = \mathcal{G}(y)) \supset \varphi(\mathcal{G}))$, where $\mathrm{sort}(\varphi) \leq n$,

φ does not have non-constructive parameters of type n other than $\mathcal{F}$.

Theory LP

We add to the language of L a predicate symbol $P_{\llcorner \overline{X}.\ \varphi \lrcorner}(z, \overline{X})$ for every formula φ of L, where all parameters of φ are in the list $\overline{X}$.

According to tradition, we will denote this symbol as $\vdash_z \varphi(\overline{X})$.

Meaning of $\vdash_z \varphi$: 'the creative subject has a proof of φ at time *z*'.

Theory LP contains all axioms of L (the schemata of the axioms are now taken for the formulas of the new language) and the following 3 axioms, in which φ is an arbitrary formula of L:

12) $(\vdash_z \varphi) \vee \neg (\vdash_z \varphi)$

13) $(\vdash_z \varphi) \supset (\vdash_{z+y} \varphi)$

14) $\exists z\, (\vdash_z \varphi) \equiv \varphi$

The intuitionistic theory LP is equiconsistent with a sub-theory I of the classical type theory TT.

In the rest of the talk I will elaborate on this result.

Theory I

It has the language of the simple set theory with arithmetic at the zero level.

The theory I has the classical predicate logic with equality and the axioms:

- Peano axioms,
- the axiom of induction over natural numbers,
- the axiom of extensionality,
- the axiom of separation: $\exists x^{n+1} \forall z^{n} (z \in x \equiv \varphi(z))$,

  where sort $(\varphi) \leq n+1$ and x^{n+1} is not a parameter of φ.

Theorem 1. There is an interpretation $\varphi \mapsto \varphi^*$ of formulas of I such that for any closed formula φ:

1) $I_s \vdash \varphi \;\;\Rightarrow\;\; LP_s \vdash \varphi^*$ (for any $s \geq 0$);

2) $I \vdash \varphi \;\;\Rightarrow\;\; LP \vdash \varphi^*$. Thus, I is consistent relative to LP.

Proof

Interpretation of I_s in $I_s^{\sim}$ (the theory I_s without the axiom of extensionality): $\varphi \mapsto \varphi^{\sim}$ (as in Friedman's interpretation).

Term $t \mapsto t'$ by replacing each x_i^n by F_i^n .

Formula $\varphi \mapsto \varphi'$: $(t \in_0 \tau)' = \exists y(\tau'(y) = S(t'))$, $(t \in_n \tau)' = \exists y(\tau'(y) = N^n(t'))$,

logical connectives are brought through.

φ^- denotes the standard negative interpretation of φ.

$\varphi^* = \varphi^{\sim\,\prime\,-}$.

Theorem 2.

1) LP_s is interpretable in I_s (for any $s \geq 0$).

2) LP is consistent relative to I.

Proof

2) follows from 1).

$s = 0$:

$P_{\llcorner x_1,..,x_r.\,\varphi \lrcorner}(z, t_1,.., t_r)$ is interpreted as $\varphi(t_1,.., t_r)$;

LP_0 is embedded into $PA = I_0$.

$s \geq 1$:

Beth-Kripke model is constructed for LP_s .

It was done formally in I_s but here we describe it informally for brevity.

It is a generalization of Beth model of van Dalen.

Beth Model for Intuitionistic Analysis (van Dalen)

Tree $(M, \precsim)$, where

$M = \omega^*$, the set of all finite sequences of natural numbers;

$\alpha \precsim \beta$ iff β is an initial segment of α (smaller nodes are higher on the tree).

Functional ξ is interpreted as ξ: $(\omega^* \times \omega) \dashrightarrow \omega$ ($\dashrightarrow$ means a partial function);
for each α, $\xi^{[\alpha]}: \omega \dashrightarrow \omega$ with properties:

(1) $\alpha \precsim \beta \Rightarrow \xi^{[\alpha]} \supseteq \xi^{[\beta]}$ (monotonicity);

(2) for any path S in the tree, $\cup\{\xi^{[\alpha]} \mid \alpha \in S\}$ is a total function.

$Q(S, \alpha)$ iff S is a path in the tree and $\alpha \in S$.

$(M, \precsim, Q)$ is the Beth scale.

The Beth model is $(\omega^*, \precsim, Q, A, V)$ where
A defines object domains and V is a valuation mapping:

for $\alpha \in \omega^*$ and atomic formula φ, $V(\alpha, \varphi)$ is t or f,

$[\alpha \precsim \beta \wedge V(\beta, \varphi) = t] \Rightarrow V(\alpha, \varphi) = t$.

The model defines forcing $\alpha \Vdash \varphi$. In particular for atomic formula φ,
$\alpha \Vdash \varphi \Leftrightarrow \forall S[Q(S, \alpha) \supset (\exists \beta \in S)(V(\beta, \varphi) = t)]$.

Notations in I_s

Ordered sequence $< x_1,.., x_r >$ has the type = max of the types of $x_1,.., x_r$.

$\mathrm{Lh}(x)$ is the length of sequence x.

$<x>_i$ is the i^{th} element of sequence x.

For a set b: $b^{(n)}$ = {all sequences of elements of b of length n};

b^* = {all finite sequences of elements of b};

$\bar{g}(n) = <g(0), ..., g(n\text{-}1)>$, "function-segment" of $g: \omega \to b$.

Sets a_k are defined by induction on k:

$a_0 = \omega$;

a_{k+1} is the set of partial functions $f: (a_k^* \times \omega) \dashrightarrow a_k$;

for each $x \in a_k^*$, $f^{[x]}: \omega \dashrightarrow \omega$ with properties:

(1) $x \lesssim y \Rightarrow f^{[x]} \supseteq f^{[y]}$ (monotonicity);

(2) for any $g: \omega \to a_k$, $\cup\{f^{[\bar{g}(n)]} \mid n \in \omega\}$ is a total function.

We follow the definitions in the book *Mathematical Intuitionism. Introduction to Proof Theory* by Dragalin.

Beth-Kripke scale

(M, ≼, Q), where

- $M = \{ x^{s-1} \mid \exists n\,(x \in a_1^{(n)} \times \ldots \times a_s^{(n)})\}$; elements of M are denoted as α, β, γ,...
- $\beta \preccurlyeq \alpha$ iff $\langle\beta\rangle_k \precsim \langle\alpha\rangle_k$ for all $k = 1, \ldots, s$;
- $Q(S, \alpha)$ iff $S = \{\langle \bar{g}_1(n), \ldots, \bar{g}_s(n)\rangle \mid n \in \omega\}$ for some functions $g_k : \omega \to a_k$, $k = 1, \ldots, s$.

Properties of Q

1) $Q(S, \alpha) \supset \alpha \in S$

2) $Q(S, \alpha) \wedge \beta \in S \supset (\alpha \preccurlyeq \beta \vee \beta \preccurlyeq \alpha)$

3) $\beta \preccurlyeq \alpha \wedge Q(S, \beta) \supset Q(S, \alpha)$

4) $Q(S, \alpha) \wedge \beta \in S \supset Q(S, \beta)$

Base for the order topology on M: sets of the form $[\alpha] = \{\beta \in M \mid \beta \preccurlyeq \alpha\}$.

Closure operator D: $DU = \{\alpha \in M \mid \forall S[Q(S, \alpha) \supset (\exists \beta \in S)(\beta \in U)]\}$.

$B = \{U \subseteq M \mid DU = U \wedge U$ is open in the order topology$\}$.
B is a pseudo-Boolean algebra with $T = M$ and $\bot = \varnothing$.

Beth-Kripke (BK) Model for LP_s in I_s

Object Domains

- ω is the domain for natural numbers;
- a_{k+1} is the domain for functionals of type k;
- the domain for constructive functionals of type k:
  $b_{k+1} = \{f \in a_{k+1} \mid f$ is independent of the first argument$\}$;
- $c_{k+1} = \{\xi \in a_{k+1} \mid \xi: (\omega \times a_k) \to a_k \wedge \forall n\, (\xi^{[n]}: a_k \to a_k$ is bijective$)\}$;

the domain for lawless functionals of type k:

$l_{k+1} = \{f \mid (\forall x \in a_k)(f^{[x]}$ is defined only for $i < \mathrm{Lh}(x)) \wedge$

$\wedge\, (\exists \xi \in c_{k+1})[f^{[x]}(i) = \xi(<i, <x>_i>)$ for any $x \in a_k$, any $i < \mathrm{Lh}(x)]$.

The BK-model defines forcing for formulas of L_s. In particular:

$\alpha \Vdash (p =_n q) \Leftrightarrow p = q$;

$\alpha \Vdash (\mathrm{Ap}^k(f, x) = y) \Leftrightarrow \forall S\, [Q(S, \alpha) \supset (\exists \beta \in S)(<<<\beta>_k, x>, y>) \in f)]$.

So $f^{[\beta]}$ depends only on $<\beta>_k$.

Forcing for formulas of LP_s

1) $\alpha \Vdash (\vdash_z \varphi) \quad \Leftrightarrow \forall S\, [Q(S, \alpha) \supset (\exists \beta \in S)(Lh(\beta) = z \wedge \beta \Vdash \varphi)]$;

2) $\alpha \Vdash (Z =_n V) \quad \Leftrightarrow \forall S\, [Q(S, \alpha) \supset (\exists \beta \in S)(Z^{[\beta]}\downarrow \wedge V^{[\beta]}\downarrow \wedge Z^{[\beta]} = V^{[\beta]})]$;

3) $\alpha \Vdash (\varphi \wedge \psi) \quad \Leftrightarrow \quad (\alpha \Vdash \varphi) \wedge (\alpha \Vdash \psi)$;

4) $\alpha \Vdash (\varphi \vee \psi) \quad \Leftrightarrow \quad \forall S\, [Q(S, \alpha) \supset (\exists \beta \in S)((\beta \Vdash \varphi) \vee (\beta \Vdash \psi)]$;

5) $\alpha \Vdash (\varphi \supset \psi) \quad \Leftrightarrow \quad (\forall \beta \preccurlyeq \alpha)((\beta \Vdash \varphi) \supset (\beta \Vdash \psi)]$;

6) $\alpha \Vdash \bot \quad \Leftrightarrow \bot$;

7) $\alpha \Vdash \neg\varphi \quad \Leftrightarrow (\forall \beta \preccurlyeq \alpha)\, \neg(\beta \Vdash \varphi)$;

8) $\alpha \Vdash \forall X\, \varphi(X) \quad \Leftrightarrow \quad (\forall f \in b)\, (\alpha \Vdash \varphi(f))$;

9) $\alpha \Vdash \exists X\, \varphi(X) \quad \Leftrightarrow \forall S\, [Q(S, \alpha) \supset (\exists \beta \in S)(\exists f \in b)\, (\beta \Vdash \varphi(f))]$;
in 8) and 9) b is a corresponding domain.

10) $(\alpha \Vdash \varphi) \wedge \beta \preccurlyeq \alpha \quad \Rightarrow \quad (\beta \Vdash \varphi)$;

11) $\forall S\, [Q(S, \alpha) \supset (\exists \beta \in S)\, (\beta \Vdash \varphi)] \quad \Leftrightarrow \quad (\alpha \Vdash \varphi)$.

$$\mathrm{LP}_s \vdash \varphi \quad \Rightarrow \quad \mathrm{I}_s \vdash (\varepsilon \Vdash \bar{\varphi}) \qquad (*)$$

where $\bar{\varphi}$ is the closure of formula φ,

$\varepsilon = \langle\langle\rangle^0, \langle\rangle^1, \ldots, \langle\rangle^{s\text{-}1}\rangle$ is the greatest element of the tree M.

The proof of (*) is by induction on the length of the derivation of φ.

• It holds for the axioms for lawless functionals because of the special construction of the tree $\mathrm{M} = \{ x^{s\text{-}1} \mid \exists n\, (x \in a_1^{(n)} \times \ldots \times a_s^{(n)}) \}$ and the interpretation of k-functionals as partial functions f: $(a_k{}^* \times \omega) \dashrightarrow a_k$.

• Axioms of choice:

7) $\forall x\, \exists!\, \mathrm{G}^n\, \varphi(x, \mathrm{G}) \supset \exists \mathrm{F}^k\, \forall x\, \varphi(x, \mathrm{F}(x)(0)^{k\text{-}n\text{-}1})$, where $k \geq \max(\mathrm{sort}(\varphi), n+1)$.

8) $\forall x\, \exists y\, \varphi(x, y) \supset \exists \mathrm{F}^n\, \forall x\, \varphi(x, \mathrm{F}(x)(0)^{n\text{-}1})$, where $n \geq \max(\mathrm{sort}(\varphi), 1)$.

(*) holds for the axioms of choice because the separation axiom in I_s has similar conditions.

• Axioms for "creative subject":

12) $(\vdash_z \varphi) \vee \neg(\vdash_z \varphi)$ 13) $(\vdash_z \varphi) \supset (\vdash_{z+y} \varphi)$ 14) $\exists z\, (\vdash_z \varphi) \equiv \varphi$

For them (*) follows from this property:

$\alpha \Vdash (\vdash_z \varphi) \Leftrightarrow \forall \mathrm{S}\, [\mathrm{Q}(\mathrm{S}, \alpha) \supset (\exists \beta \in \mathrm{S})(\mathrm{Lh}(\beta) = z \wedge \beta \Vdash \varphi)]$.

Some Corollaries

From a result of Tarski it follows that the truth predicate for I_s can be defined in I_{s+1} ($s \geq 0$), so the consistency I_s of is derived in I_{s+1} .

- I_{s+1} is stronger than I_s .

I_0 is arithmetic PA,
I_1 is second-order arithmetic Ar2.

- I is stronger than Ar2.

Since I_s and LP_s are equiconsistent:

- LP_{s+1} is stronger than LP_s .
- LP is stronger than Ar2.

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