Distributive Proper Forcing Axiom

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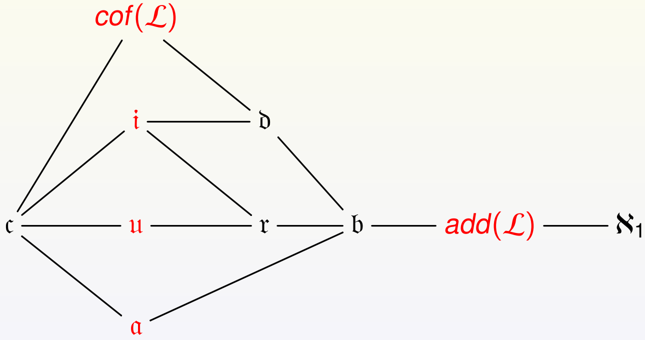
- *Question: MRP* $\rightarrow$ *add*$(\mathcal{L}) = \mathfrak{c}$?

Cardinal Invariants



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- If $\mathbb{P}$ is proper, then $\mathbb{P}$ is distributive iff it adds no new reals.
- *DPFA* (Distributive Proper Forcing Axiom) is the following statement: If $\mathbb{P}$ is distributive and proper, $\mathfrak{D}$ is a collection of dense subsets of $\mathbb{P}$, $|\mathfrak{D}| = \aleph_1$, then there is a filter $G \subseteq \mathbb{P}$ s.t. $\forall D \in \mathfrak{D}, G \cap D \neq \emptyset$.

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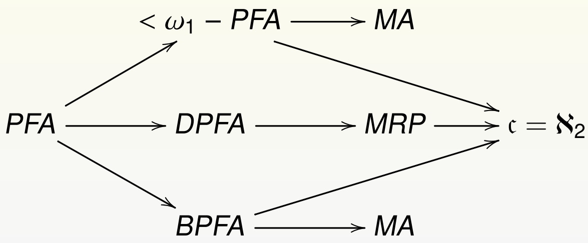
PFA and its Fragments



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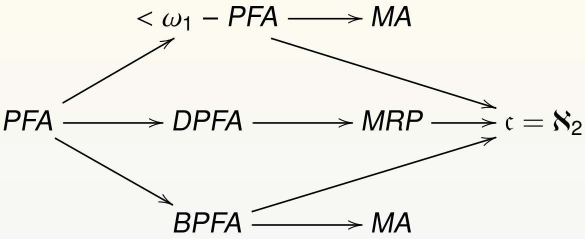

Question: What is the relationship among these fragments of *PFA* ?

The Main Theorem

Theorem [2010]

If it is consistent that a supercompact cardinal exists, then it is consistent that *DPFA* holds and $\mathfrak{a} = \mathfrak{u} = \mathfrak{i} = cof(\mathcal{L}) = \aleph_1 < \mathfrak{c}$.

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Corollary

DPFA, *BPFA* and $< \omega_1 - PFA$ are mutually independent.

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- The witnesses found in step 1 survive in step 2.

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- At limit stages, take finite support.
- $\mathbb{A} = \mathbb{A}_{\omega_1}$.

Properties

- If $a \in \mathbb{V}^{\mathbb{A}_\beta} \cap [\omega]^\omega$ satisfying $\forall \alpha < \beta, |a \cap a_\alpha| < \omega$, then $\mathbb{V}^{\mathbb{A}_{\beta+1}} \models |a \cap a_\beta| = \omega$.

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- $\mathbb{V}^{\mathbb{A}} \vDash \{a_\alpha : \alpha < \omega_1\}$ is a MAD family.

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- The limit $\mathbb{D}$ is proper, but not distributive!

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- $q \leq p$, q is $(N, \mathbb{Q})$-generic, $q \Vdash a_\xi$ is $(N[\dot{G} \cap N], B_\xi)$-generic.

Why "Nice"

- If $\mathbb{V}^{\mathbb{A}} \vDash \mathbb{Q}$ is nice, then $\mathbb{V}^{\mathbb{A}*\mathbb{Q}} \vDash \{a_\alpha : \alpha < \omega_1\}$ is MAD.

Why "Nice"

- If $\mathbb{V}^{\mathbb{A}} \vDash \mathbb{Q}$ is nice, then $\mathbb{V}^{\mathbb{A}*\mathbb{Q}} \vDash \{a_\alpha : \alpha < \omega_1\}$ is MAD.
- Proof: Otherwise, in $\mathbb{V}^{\mathbb{A}}$, there will be some $p \in \mathbb{Q}$, and some $\mathbb{Q}$-name τ such that $p \Vdash \tau$ witnesses that $\{a_\alpha : \alpha < \omega_1\}$ is not maximal. Let $N \prec H_\lambda$ be such that $\{\tau, p, Q\} \subset N$ and $\xi \in \omega_1 \setminus N$, such that a_ξ is (N, B_ξ)-generic. By the definition of niceness, $\exists q \leq p$, q is $(N, \mathbb{Q})$-generic and $q \Vdash a_\xi$ is $(N[\dot{G} \cap N], B_\xi)$-generic. Then $q \Vdash \exists \alpha \leq \xi \ \ |\tau \cap a_\alpha| = \omega$, a contradiction.

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- Proof: Notice that each B_ξ is countable.

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- Proof: By induction on length of the iteration.

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- $\mathbb{V}^{\mathbb{A}*\mathbb{D}} \models DPFA \wedge \mathfrak{a} = \aleph_1$.

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- Dense open set property of $\omega^{<\omega}$. etc.

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Thank you!