Sahlqvist fixed point formulas

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joint work with
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Introduction

Sahlqvist theory is a core area of modal logic.

Sahlqvist modal formulas originate with Sahlqvist (1973).

They are a syntactically-defined class of modal formulas.

1. widely occurring
2. have computable first-order frame correspondents
3. canonical

Any Sahlqvist-axiomatisable logic is sound and complete for the class of Kripke frames defined by the frame correspondents of the axioms.

Aim of talk: sketchy description of how to extend Sahlqvist formulas to mu-calculus (modal fixed point logic), keeping (1) and (2) in some sense. (I will discuss canonicity a little at the end.)

Modal logic (notation)

Primitive connectives are $\wedge, \vee, \neg, \Box, \Diamond$.
$\varphi \to \psi$ abbreviates $\neg\varphi \vee \psi$.

A modal formula is *positive* if it does not involve $\neg$,
and *negative* if it is of the form $\neg\pi$ for positive π.
$\Box^d\varphi = \underbrace{\Box\Box\ldots\Box}_{d \text{ times}}\varphi$, for $d \geq 0$.

Kripke frames: $\mathcal{F} = (W, R)$.
Assignments: $h : \{\text{atoms}\} \to \wp(W)$.

Semantics: $\mathcal{F}, h, w \models \varphi$ defined as usual.
$[\![\varphi]\!]_h = \{w \in W : \mathcal{F}, h, w \models \varphi\}$.

Classical (modal) Sahlqvist formulas

Can define Sahlqvist formulas φ by BNF:

$$\varphi ::= \neg\Box^d p \mid \pi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \Box\varphi$$

where p is an atom, $d \geq 0$, and π is a positive formula.

Equivalently: formulas of the form $\neg\sigma(\beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n)$ where

- the *skeleton* $\sigma(b_1, \ldots, b_m, q_1, \ldots, q_n)$ involves only $\vee, \wedge, \Diamond$
- $\beta_1, \ldots, \beta_m$ are *boxed atoms* — of the form $\Box^d p$ (for some $d \geq 0$)
- $\gamma_1, \ldots, \gamma_n$ are *negative* formulas.

Examples

$\Box p \to p$ ($= \neg\Box p \vee p$, equivalent to $\neg(\Box p \wedge \neg p)$ — skeleton is $b \wedge q$)
$\Diamond\Box p \to \Box\Diamond p$ (Church–Rosser, $\equiv \neg(\Diamond\Box p \wedge \neg\Box\Diamond p)$. Skeleton is $\Diamond b \wedge q$.)

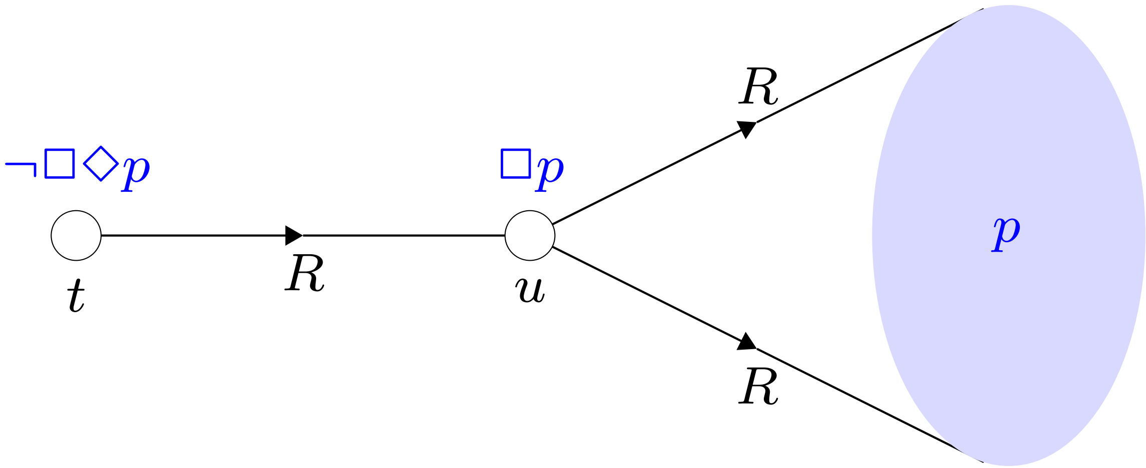
Non-examples (not equivalent to Sahlqvist formulas)

Löb's axiom, $\Box(\Box p \to p) \to \Box p$
McKinsey's formula, $\Box\Diamond p \to \Diamond\Box p$

Example of Sahlqvist correspondence: Church–Rosser

Assume $\chi = \Diamond\Box p \to \Box\Diamond p$ is *not* valid in some Kripke frame $\mathcal{F} = (W, R)$ at some $t \in W$ (in symbols, $\mathcal{F}, t \not\models \chi$).

This says that there are an assignment h : {atoms} $\to \wp(W)$,
and $u \in W$, with: $R(t, u)$, $\mathcal{F}, h, u \models \Box p$, and $\mathcal{F}, h, t \models \neg\Box\Diamond p$:



We can replace h by the *minimal assignment* h° satisfying $\mathcal{F}, h, u \models \Box p$.
Plainly, $h^\circ(p) = \{x \in W : R(u, x)\}$ — first-order-definable.

Obtaining first-order correspondent

So $\mathcal{F}, t \not\models \chi$ is equivalent to $\exists u(Rtu \wedge \mathcal{F}, h^\circ, t \models \neg\Box\Diamond p)$.

Using 'standard translation', we can express this in first-order logic in the signature of frames:

$$\mathcal{F} \models \exists u(Rtu \wedge \neg\forall v(Rtv \to \exists w(Rvw \wedge \underbrace{Ruw}_{w \in h^\circ(p)}))).$$

Conclude

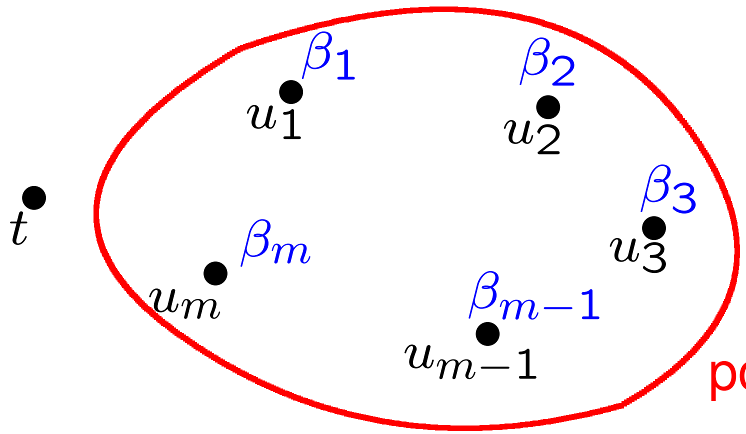
- $\mathcal{F}, t \models \chi$ iff $\mathcal{F} \models \forall u(Rtu \to \forall v(Rtv \to \exists w(Rvw \wedge Ruw)))$,
- χ is valid in $\mathcal{F}$ iff $\mathcal{F} \models \forall tu(Rtu \to \forall v(Rtv \to \exists w(Rvw \wedge Ruw)))$ — first-order correspondent.

What does this argument really use?

For an arbitrary Sahlqvist formula $\neg\sigma(\beta_1, \dots, \beta_m, \gamma_1, \dots, \gamma_n)$, the above argument uses that:

1. $\sigma(b_1, \dots, b_m, q_1, \dots, q_n) \equiv$
   $\exists u_1, \dots, u_m(\sigma(\{u_1\}, \dots, \{u_m\}, q_1, \dots, q_n) \wedge \bigwedge_{1 \le i \le m} u_i \models b_i)$.

   Then we can extract worlds $u_1, \dots, u_m$ where the boxed atoms hold:



(1) says that σ is *completely additive in* $b_1, \dots, b_m$.
A formula $\varphi(p)$ is *completely additive in* p if $[\![\varphi(\bigcup_i S_i)]\!] = \bigcup_i [\![\varphi(S_i)]\!]$ for any sets $S_i \subseteq W$ $(i \in I)$.

What else does it use?

2. When a boxed atom $\beta(p) = \Box^d p$ is true at a world, there is a *minimal assignment* making it true.

   Complete multiplicativity of $\beta(p)$ is sufficient for this: that is,
   $[\![\beta(\bigcap_i S_i)]\!] = \bigcap_i [\![\beta(S_i)]\!]$ for any sets $S_i \subseteq W$.
   Then, the minimal assignment making β true is just the intersection of *all* assignments making it true.

3. The minimal assignment h° is first-order *definable.*

4. Each negative formula is *antitonic* in all its atoms, and σ is *monotonic* in $q_1, \ldots, q_n$ (so replacing h by h° preserves the negative formulas).

These are the principles we use.
So can we generalise the argument?

PIA formulas [van Benthem, JSL 2005]

These generalise the boxed atoms $\Box^d p$.

Modal PIA formulas can be defined by

$$\beta ::= p \mid \beta_1 \wedge \beta_2 \mid \pi \to \beta \mid \Box \beta$$

where p is an atom, and π is positive.
(JvB originally restricted to $\beta(p)$ only; restriction no longer needed.)

Examples: boxed atoms $\Box^n p$, antecedent of Löb's axiom: $\Box(\Box p \to p)$.

Any PIA formula is completely multiplicative.
So when true at a world, it has a minimal assignment making it true.

This *minimal assignment is definable* — not necessarily in first-order logic, but *in FO+LFP.*

Generalised modal Sahlqvist formulas (van Benthem 2005)

So: generalise Sahlqvist formulas φ by *replacing '$\Box^n p$' by 'PIA':*

$$\varphi ::= \neg\beta \mid \pi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \Box\varphi$$

where β is PIA and π is positive.
Frame correspondents will now be in FO+LFP (first-order logic with least fixed points).

Examples *Löb's axiom,* $\Box(\Box p \to p) \to \Box p$, is equivalent to

$$\neg \underbrace{\Box(\Box p \to p)}_{\text{PIA}} \vee \underbrace{\Box p}_{\text{positive}}$$

Can show $\mathcal{F}, t \models \Box(\Box p \to p) \to \Box p$ iff:
(1) R is transitive from t, and (2) R is conversely well-founded at t.
This is definable in FO+LFP.

McKinsey's formula has no FO+LFP frame correspondent (vB–Goranko).

Towards the modal mu-calculus

Recall the *mu-calculus syntax:*

$$\varphi ::= p \mid x \mid \neg\varphi \mid \varphi \vee \varphi' \mid \varphi \wedge \varphi' \mid \Diamond\varphi \mid \Box\varphi \mid \mu x\varphi \mid \nu x\varphi$$

where x is a fixed point variable and occurs only positively in φ.

Semantics: $\mathcal{F}, h, t \models \mu x\varphi$ iff t is in the least fixed point of the map $(X \mapsto [\![\varphi]\!]_{h[x \mapsto X]})$. ($\nu x\varphi$ similar, using greatest fixed point.)

Eg. $\mu x(p \vee \Diamond x)$ defines $\Diamond^* p$ (reflexive transitive closure of $\Diamond$).
Mu-calculus formulas have standard translations into FO+LFP.

If we are happy with frame correspondents in FO+LFP, why not generalise Sahlqvist formulas to the *modal mu-calculus?*

Would give a wider class of formulas with FO+LFP-frame correspondents.

We can, if we can find a nice class of *completely additive mu-calculus formulas.*

$\mathcal{Q}$-skeletons — main technical device

Definition 1 *Let $\mathcal{Q}$ be a set of atoms. The $\mathcal{Q}$-skeletons are defined by:*

$$\sigma ::= p \mid x \mid \sigma \vee \sigma' \mid \Diamond\sigma \mid \mu x \sigma \mid \sigma \wedge \tau$$

where τ is a sentence with no atoms from $\mathcal{Q}$.

Lemma 2 (complete additivity) *Let σ be a $\mathcal{Q}$-skeleton, and $\mathcal{H}$ a non-empty set of assignments (into some frame) that agree on all atoms not in $\mathcal{Q}$.*
Let g be the assignment given by $g(\xi) = \bigcup\{h(\xi) : h \in \mathcal{H}\}$ for each ξ.
Then

$$[\![\sigma]\!]_g = \bigcup_{h \in \mathcal{H}} [\![\sigma]\!]_h.$$

Proof. Induction on σ — exercise. □

There are earlier related results by G. Fontaine. This lemma covers the skeletons of Sahlqvist formulas, and (dually) PIA formulas as well.

The outcome: Sahlqvist fixed point formulas

PIA mu-formulas:

$$\beta ::= p \mid x \mid \beta_1 \wedge \beta_2 \mid \pi \to \beta \mid \Box\beta \mid \nu x \beta$$

where p is an atom, x a fixed point variable, and π a positive sentence.

Sahlqvist mu-formulas:

$$\sigma ::= \neg\beta \mid \pi \mid x \mid \sigma_1 \wedge \sigma_2 \mid \sigma_1 \oplus \sigma_2 \mid \Box\sigma \mid \nu x \sigma$$

where β is a PIA sentence, π a positive sentence, x a f.p. variable, and

$$\sigma_1 \oplus \sigma_2 = \begin{cases} \sigma_1 \vee \sigma_2, & \text{if } \sigma_1, \sigma_2 \text{ are both sentences,} \\ & \text{or one of them is a positive sentence,} \\ \text{undefined,} & \text{otherwise.} \end{cases}$$

Theorem 3 (JvB, NB, IH, 2011) *Any Sahlqvist mu-sentence has an (easily computable) frame correspondent in FO+LFP.*

Canonicity (joint work with N. Bezhanishvili)

Using a weaker definition, Sahlqvist mu-formulas have been shown to be canonical, giving a completeness theorem.

One needs to define *modal mu-algebras* that are closed under μ, ν.
We used *admissible semantics* (μ, ν relativised to subset of the algebra).

The same (weaker) Sahlqvist mu-formulas are preserved by
Monk completions of conjugated algebras
(extends result of Givant–Venema for modal Sahlqvist formulas).

Next steps?

1. Find interesting subfragments (or extensions) of Sahlqvist mu-formulas (eg Sahlqvist PDL-formulas).

2. Are known generalisations of *modal* Sahlqvist formulas covered? (Eg Conradie–Goranko–Vakarelov)

3. Does the SQEMA algorithm of said workers extend to Sahlqvist mu-formulas? Does it go further?

4. What can be said about canonicity of Sahlqvist mu-formulas?

5. Correspondence/canonicity of strong Sahlqvist mu-formulas in ‘admissible semantics?

6. Generally: find more extensions of classical modal results to the mu-calculus!!

References (some are at `www.doc.ic.ac.uk/~imh/`)

- Johan van Benthem, *Minimal predicates, fixed-points, and definability,* J. Symbolic Logic, 70 (2005), 696–712.
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Thank you for your patience.