Nonempty open intervals of the effectively closed Muchnik degrees

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In the last decade, the distributive lattice $\mathcal{P}_w$ is well studied in computability theory.

Here, $\mathcal{P}_w$ is the distributive lattice of weak (or Muchnik) degrees of nonempty effectively closed sets in Cantor space 2^ω.

A major open question is whether $\mathcal{P}_w$ is dense.

(I.e., for any $a, b \in \mathcal{P}_w$, is there $c \in \mathcal{P}_w$ with $a < c < b$?)

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The aim of this talk is to introduce a partial solution to the question.

$\omega^\omega := \{f \mid f : \omega \to \omega\}$, Baire space.
$2^\omega := \{f \mid f : \omega \to \{0,1\}\}$, Cantor space.

Let $P, Q \subset \omega^\omega$.

P is weakly (or Muchnik) reducible to Q, denoted by $P \leq_w Q$,
$:\Leftrightarrow (\forall g \in Q)(\exists f \in P)[f \leq_T g]$.

P is Π^0_n iff
there is a computable relation $R \subset \omega^n \times \omega^\omega$ such that
$f \in P \iff (\forall x_0)(\exists x_1)(\forall x_2)(\exists x_3)\cdots(Qx_{n-1})[R(\vec{x}, f)]$.

$\mathcal{P}_w :=$(the nonempty Π^0_1 subsets of 2^ω)$/\equiv_w$,
where $P \equiv_w Q$ (P is weakly equivalent to Q)
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Main theorem

Let nonempty Π^0_1 sets $P, Q \subset 2^\omega$ satisfy $P <_w Q$.
(Here, $P <_w Q$ iff $P \leq_w Q$ & $P \not\geq_w Q$.)

$(\exists \Pi^0_4 S \subset P)(\exists p \in S)[\{p\} \not\geq_w Q \ \& \ \neg(\exists q \in S)[p >_T q]]$
$\Longrightarrow (\exists \text{nonempty } \Pi^0_1 \ R \subset 2^\omega)[P <_w R <_w Q]$.

Let's see the sketch of a proof!

Embedding Lemma (Simpson):
$(\forall \Pi^0_1 Q \neq \emptyset \text{ and } \subset 2^\omega)(\forall \Pi^0_2 R \subset \omega^\omega)(\exists \Pi^0_1 R' \neq \emptyset \text{ and } \subset \omega^\omega)$
$[\ R' \equiv_w R \cup Q\ ]$.

For nonempty Π^0_1 sets $P <_w Q$, find Π^0_2 R such that $P <_w R \not\geq_w Q$.
Then, $P <_w R' \equiv_w R \cup Q <_w Q$.

WANTED (For $P <_w Q$)

Π^0_2 R with $P <_w R \not\geq_w Q$.

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For $f_0, f_1 \in \omega^\omega$, let $f_0 \oplus f_1(2x+i) := f_i(x)$.
For $R_0, R_1 \subset \omega^\omega$, let $R_0 \vee R_1 := \{f_0 \oplus f_1 \mid f_i \in R_i\}$.

If $R^f \subset \omega^\omega$ is $\Pi^0_2(f)$ uniformly in $f \in P$, then $\bigcup_{f \in P}(\{f\} \vee R^f)$ is Π^0_2.

$\because \bigcup_{f \in P}(\{f\} \vee R^f) = \{f \oplus g \mid f \in P \ \&\ g \in R^f\}$.

We want to construct R^f's with the following property:
$P <_w \bigcup_{f \in P}(\{f\} \vee R^f) \not\geq_w Q$.

We can do this under our assumption!

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Let P, Q be nonempty Π^0_1 sets $P <_w Q$ and let $S \subset P$ be Π^0_4 with $(\exists p \in S)[\{p\} \not\geq_w Q \ \& \ \neg(\exists q \in S)[p >_T q]]$.

Using the priority method with infinite injury, we can construct $\Pi^0_2(f)$ set $R^f \subset \omega^\omega$ uniformly in f for all $f \in P$ such that

$$f \in S \ \& \ \{f\} \not\geq_w Q \Rightarrow \{f\} \not\geq_w R^f \ \& \ \{f\} \vee R^f \not\geq_w Q,$$
$$f \notin S \ \& \ \{f\} \not\geq_w Q \Rightarrow R^f \geq_w Q.$$

Then, $P <_w \bigcup_{f \in P}(\{f\} \vee R^f) \not\geq_w Q$.
$\because \ \{p\} \not\geq_w \{f\} \vee R^f$ since $p \geq_T f \Rightarrow \{p\} \equiv_w \{f\} \not\geq_w R^f$.
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Let P, Q be nonempty Π^0_1 sets with $P <_w Q$.

Main theorem

$$(\exists \Pi^0_4 S \subset P)(\exists p \in S)[\{p\} \not\geq_w Q \ \& \ \neg(\exists q \in S)[p >_T q]] \quad \cdots\cdots (\star)$$

$$\Longrightarrow (\exists \text{nonempty } \Pi^0_1 R \subset 2^\omega)[P <_w R <_w Q].$$

We see examples that satisfy $(\star)$.

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The following examples satisfy $(\star)$ as $S = P$.

- P consists of pairwise Turing incomparable elements.
- P has at most finitely many elements f with $\{f\} \not\geq_w Q$.

Let P, Q be nonempty Π^0_1 sets with $P <_w Q$.

Main theorem

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Corollary

$(\exists \Delta^0_4 p \in P)[\{p\} \not\geq_w Q]$
$\Longrightarrow (\exists \text{nonempty } \Pi^0_1 R \subset 2^\omega)[P <_w R <_w Q]$.

$\because$ Let R, R' be Π^0_4 relations with $p(x) = 1$ iff $R(x)$ iff $\neg R'(x)$.
$f \in \{p\} \Leftrightarrow (\forall x)[[p(x) = 1 \Rightarrow R(x)] \ \&\ [p(x) = 0 \Rightarrow R'(x)]]$.

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$(\exists\Delta^0_4 p \in P)[\{p\} \not\geq_w Q] \cdots\cdots (\triangle)$
$\Longrightarrow (\exists \text{nonempty } \Pi^0_1 R \subset 2^\omega)[P <_w R <_w Q]$.

By this corollary, we can deduce stronger results than one proved by Stephen Binns:
P is small & $\neg(\exists\Pi^0_1 Q' \subset Q)[Q' \text{ is nonempty very small}]$
$\Longrightarrow (\exists \text{nonempty } \Pi^0_1 R \subset 2^\omega)[P <_w R <_w Q]$.

Each of the following two conditions implies $(\triangle)$:
- P is small & $\neg(\exists\Pi^0_1 Q' \subset Q)[Q' \text{ is nonempty small}]$,
- P is very small & $\neg(\exists\Pi^0_1 Q' \subset Q)[Q' \text{ is nonempty very small}]$.

Indeed, we have $\Delta^0_3 p \in P$ witnessing $P \not\geq_w Q$.

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Thank you for your attention.