Variants of Constructions of Noncuppable Degrees

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Computably enumerable sets and degrees

- Halting problem and Post problem, early structural results on c.e. degrees
- Shoenfield Conjecture — Lachlan and Yates' work
- Cuppable degrees and cappable degrees
- Noncuppable degrees (Yates) and noncappable degrees (Lachlan)
- Harrington' work:
  - Every high c.e. degree bounds a high noncuppable degree.
  - cups or caps — all noncuppable degrees are cappable.

Low-Cuppable=Noncappable

- Low_n cuppable degrees, LC_n
- (Ambos-Spies, Jockusch, Shore and Soare 1984)
  Low-cuppable degrees, noncappable, promptly simple degrees coincide with each other.
- (Li, Wu and Zhang 2000)
  There is a c.e. degree, cappable and low_2-cuppable. Thus, LC is a proper subset of LC_2.
- Question: Is it true that for each n, there is a c.e. degree low_{n+1}-cuppable, but not low_n-cuppable?

Locally noncappable degrees

- Definition: (Seetapun)
  A nonzero c.e. degree **a** is locally noncappable if there is a c.e. degree **c** above **a** such that no nonzero c.e. degree below **c** can form a minimal pair with **a**.

  We say that **c** witnesses that **a** is locally noncappable.

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  As a corollary, there is no maximal nonbounding degree, as when **a** is a nonbounding degrees, then so it **c**.

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- Theorem: (Stephan and W.)
  The witness **c** can be high$_2$. So
  - There is a high$_2$ nonbounding degree (Downey, Lempp and Shore).
  - There is a high$_2$ plus-cupping degree, in terms of Harrington (Li).
  - There is a high$_2$ degree bounding no bases of Slaman triples (Leonardi).

- (Fang, Wang and W.) For any nonzero and incomplete c.e. degree **a**, there are two c.e. degrees $\mathbf{c}_1$ and $\mathbf{c}_2$ above **a** such that $\mathbf{c}_1 \cup \mathbf{c}_2$ is high. As a corollary, we can have that
  - nonbounding degrees do not form an ideal (Li and Wang).
  - plus-cupping degrees do not form an ideal (Li and Wang).

  Li-Wang's two results were proved by two separated arguments.

High degrees, i.e. $H' \equiv_T K'$

- A function is dominant if it dominates all computable functions.

- (Tennenbaum 1963, Martin 1966) If A is maximal then p_A is dominant.

- (Martin) A set X is high iff there is a dominant function $f \leq_T X$.

- (Martin) For c.e. degrees, **a** say, **a** has high degree iff **a** contains a maximal c.e. set A iff p_A is dominant.
  -
  - Cooper: Every high c.e. degree bounds a minimal pair.
  - Shore and Slaman: every high c.e. degree bounds a Slaman triple.

Superhigh sets, i.e. $H' \equiv_{tt} K'$

- Superhigh and superlow sets are proposed by Mohrherr in her thesis.
  1. incomplete superhigh c.e. degrees.
  2. high, but not superhigh, c.e. degrees.

- Ng (Shore, unpublished): a minimal pair of superhigh degrees

- Jockusch and Mohrherr: the diamond lattice embedding into the c.e. *tt*-degrees: 0 and 1 with two atoms (super)low.
- Cenzer, Johanna, Liu and W.: two atoms can be superhigh.

Almost everywhere dominating degrees

Definition:

(Dobrinen and Simpson)
A set A is almost everywhere dominating (a.e.d. for short) if for almost all sets X and all functions $g \leq_T X$, there is a function $f \leq_T A$ such that f dominates g.

- Kurtz: $\emptyset'$ has this property.
- Cholak, Greenberg and Miller (2004): an incomplete c.e. set with almost everywhere dominating property.
- Simpson 2007: almost everywhere dominating sets are superhigh.
- Simpson 2007: a superhigh c.e. set without almost everywhere dominating property.

- Question: Are there a minimal pair of two c.e. sets with almost everywhere dominating property?

Superlow sets and jump traceability

- Superlow sets can have joint $\emptyset'$, and K-trivial sets are superlow.
- Over c.e. sets, “superlow” and “jump traceable”coincide.

  A set A is *h-jump traceable* for a computable order h if there is a uniformly c.e. sets

  $$T_0, T_1, T_2, \cdots,$$

  such that $|T_n| \leq h(n)$ and that if $\Phi_e^{(}e)$ converges, $\Phi_e^{(}e)$ is in T_e for each e.

  A set A is jump traceable if it is h-jump traceable for some computable function h.

- (Diamondstone) There is a low-cuppable degree, which is not cuppable to $\emptyset''$ by any superlow c.e. set.

wtt-reduction and array computable degrees

- (Downey, Jockusch and Stob) A degree is array noncomputable (anr) if and only if there is a function $f \leq_{wtt} \emptyset'$ that dominates all **a**-computable functions. A degree is array computable if it is not anr.
- Fact: all array computable degrees are low_2.
- (Downey, Jockusch and Stob) There is a low c.e. anc degree. So array computable c.e. degrees form a proper subclass of low_2 c.e. degrees.

Notion on c.e. traceability and Ishmukametov's work

A set A, and the degree of A, are c.e. traceable if there is a computable function p (called a bound) such that for each function $f \leq_T A$, there is a computable function h (called a trace of f) such that for all n,

1. $|W_{h(n)}| \leq p(n)$ and
2. $f(n) \in W_{h(n)}$.

- A c.e. degree is array computable if and only if it is c.e. traceable if and only if it has a strong minimal cover.

- (Ishmukametov) If the c.e. degrees are definable in the global structure of degrees, then so are the anc degrees.

Similarly, we can say that a wak truth-table degree **a** is c.e. traceable if there is a computable order h such that for each function $f \leq_{\text{wtt}} \mathbf{a}$, there is a computable collection $\{W_{g(x)} : x \in \omega\}$ with

1. $|W_{g(n)}| \leq h(n)$ and
2. $f(n) \in W_{h(n)}$.

Theorem (Franklin, Greenberg, Stephan and W.)

For a set A, the following are equivalent:

1. The wtt degree of A is c.e. traceable;
2. A is wtt-reducible to a Schnorr trivial set;
3. A is anti-complex.
   That is, for every computable order h, $C(A \upharpoonright h(n)) \leq n$ for almost all n.

Slaman's Cupping Theorem

Theorem

There are two incomparable degrees **a** and **c** such that **c** cups each c.e. degree below **a**, but not below **c**, to $\mathbf{0}'$.

This cupping theorem gives the existence of a noncuppable degree.

- If **a** bounds a noncuppable degree, then it is also below **c**.
- Otherwise, **c** should bounds a noncuppable degree.

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A direct construction of noncuppable degrees

Construct a c.e. set A such that

- A is incomputable;
- For any c.e. set W, if $A \oplus W_e$ computes $\emptyset'$, then W_e also computes $\emptyset'$.

We build a c.e. set P and a p.c. functional Δ_e such that

- If $P = \Phi_e^{AW_e}$, then $\emptyset' = \Delta_e^{W_e}$.

- Interactions between strategies.

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Cuppable or noncuppable?

Theorem (Li, W. and Yang)

There are two cuppable c.e. degrees **a** and **b** such that $\mathbf{0}'$ is the only c.e. degree cupping both to $\mathbf{0}'$.

We build c.e. sets A, B and also C, D, F, P such that

- $A \oplus C \geq_T \emptyset'$;
- $B \oplus D \geq_T \emptyset'$;
- $F \not\leq_T C$, $F \not\leq_T D$;

- If $P = \Phi_e^{AW_e} = \Psi_e^{BW_e}$, then $\emptyset' = \Delta_e^{W_e}$.

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This result shows the minimal pair exists in the quotient $\mathbf{R}/\mathcal{N}Cup$.

Recall that the quotient $\mathbf{R}/\mathcal{M}$ does not have minimal pairs.

Back to LC_n

Theorem (Greenberg, Ng and W.)

There is a cuppable c.e. degrees $\mathbf{a}$ such that all the c.e. degrees cupping $\mathbf{a}$ to $\mathbf{0}'$ are high.

This result says that $\bigcup_n LC_n$ does not exhaust all cuppable degrees, refuting a claim of Li in 2000 that any cuppable degree is low_3-cuppable.

We build c.e. sets A, C and also F, P such that

- $A \oplus C \geq_T \emptyset'$;
- $F \not\leq_T C$;

- If $P = \Phi_e^{AW_e}$, then there is a p.c. functional Δ_e such that for all i,

$$TOT(i) = \lim_x \Delta_e^{W_e}(i, x).$$

- Again, there is a direct conflict between diagonalization and the highness strategies.

Thanks!