Definability in metric structures

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Metric Structures

- A (bounded) *metric structure* is a (bounded) complete metric space (M, d), together with distinguished
  1. elements,
  2. functions (mapping M^n into M for various n), and
  3. predicates (mapping M^n into a bounded interval in $\mathbb{R}$ for various n).
- Each function and predicate is required to be uniformly continuous.
- For the sake of simplicity, we suppose that the metric is bounded by 1 and the predicates all take values in $[0, 1]$.

Examples of Metric Structures

1. If $\mathcal{M}$ is a structure from classical model theory, then we can consider $\mathcal{M}$ as a metric structure by equipping it with the discrete metric. If $P \subseteq M^n$ is a distinguished predicate, then we consider it as a mapping $P : M^n \to \{0, 1\} \subseteq [0, 1]$ by

$$P(a) = 0 \text{ if and only if } \mathcal{M} \models P(a).$$

2. Suppose X is a Banach space with unit ball B. Then $(B, 0_X, \|\cdot\|, (f_{\alpha,\beta})_{\alpha,\beta})$ is a metric structure, where $f_{\alpha,\beta} : B^2 \to B$ is given by $f(x, y) = \alpha \cdot x + \beta \cdot y$ for all scalars α and β with $|\alpha| + |\beta| \leq 1$.
3. If H is a Hilbert space with unit ball B, then $(B, 0_H, \|\cdot\|, \langle\cdot,\cdot\rangle, (f_{\alpha,\beta})_{\alpha,\beta})$ is a metric structure.

Bounded Continuous Signatures

- As in classical logic, a signature L for continuous logic consists of constant symbols, function symbols, and predicate symbols, the latter two coming also with arities.
- New to continuous logic: For every function symbol F, the signature must specify a *modulus of uniform continuity* Δ_F, which is just a function $\Delta_F : (0,1] \to (0,1]$. Likewise, a modulus of uniform continuity is specified for each predicate symbol.
- The metric d is included as a (logical) predicate in analogy with $=$ in classical logic.

L-structures

An *L-structure* is a metric structure $\mathcal{M}$ whose distinguished constants, functions, and predicates are interpretations of the corresponding symbols in L. Moreover, the uniform continuity of the functions and predicates is witnessed by the moduli of uniform continuity specified by L.

e.g. If P is a unary predicate symbol, then for all $\epsilon > 0$ and all $x, y \in M$, we have:

$$d(x, y) < \Delta_P(\epsilon) \Rightarrow |P^{\mathcal{M}}(x) - P^{\mathcal{M}}(y)| \leq \epsilon.$$

Formulae

- Atomic formulae are now of the form $d(t_1, t_2)$ and $P(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are terms and P is a predicate symbol.
- We allow all continuous functions $[0,1]^n \to [0,1]$ as n-ary connectives.
- If φ is a formula, then so is $\sup_x \varphi$ and $\inf_x \varphi$. (sup $= \forall$ and inf $= \exists$)

Definable predicates

- If M is a metric structure and $\varphi(x)$ is a formula, where $|x| = n$, then the interpretation of φ in M is a uniformly continuous function $\varphi^M : M^n \to [0,1]$.
- For the purposes of definability, formulae are not expressive enough. Instead, we broaden our perspective to include *definable predicates*.
- If $A \subseteq M$, then a uniformly continuous function $P : M^n \to [0,1]$ is *definable in M over A* if there is a sequence $(\varphi_n(x))$ of formulae with parameters from A such that the sequence (φ_n^M) converges uniformly to P.

Definable functions

- $f : M^n \to M$ is *A-definable* if and only if the map $(x, y) \mapsto d(f(x), y) : M^{n+1} \to [0, 1]$ is an A-definable predicate.
- A new complication: Definable sets and functions may now use *countably* many parameters in their definitions. If the metric structure is separable and the parameterset used in the definition is dense, then this can prove to be troublesome.
- Given any elementary extension $N \succeq M$, there is a natural extension of f to an A-definable function $\tilde{f} : N^n \to N$.

Definability takes a backseat

- There are notions of stability, simplicity, rosiness, NIP,... in the metric context. These notions have been heavily developed with an eye towards applications.
- However, old-school model theory in the form of definability has not really been pursued. In particular, the question: "Given a metric structure M, what are the sets and functions definable in M?" has not received much attention. This is the question that we will focus on in this talk.

Definable closure

Definition

Given an L-structure M, a parameterset $A \subseteq M$, and $b \in M$, we say that b is *in the definable closure of* A, written $b \in \mathrm{dcl}(A)$, if the predicate $x \mapsto d(x, b) : M \to [0, 1]$ is an A-definable predicate.

Facts

Let M be a structure, $A \subseteq M$, and $b \in M$.

- If $b \in \mathrm{dcl}(A)$, then there is a *countable* $A_0 \subseteq A$ such that $b \in \mathrm{dcl}(A_0)$.
- If M is ω_1-saturated and A is countable, then $b \in \mathrm{dcl}(A)$ if and only if $\sigma(b) = b$ for each $\sigma \in \mathrm{Aut}(M/A)$.
- $\bar{A} \subseteq \mathrm{dcl}(A)$ ($\bar{A}$=metric closure of A)
- If $f : M^n \to M$ is an A-definable function, then for each $x \in M^n$, we have $f(x) \in \mathrm{dcl}(A \cup \{x_1, \ldots, x_n\})$.

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The Urysohn sphere

Definition

The *Urysohn sphere* $\mathfrak{U}$ is the unique, up to isometry, Polish metric space of diameter ≤ 1 satisfying the following two properties:

- universality: any Polish metric space of diameter ≤ 1 admits an isometric embedding in $\mathfrak{U}$;
- ultrahomogeneity: any isometry between finite subspaces of $\mathfrak{U}$ can be extended to a self-isometry of $\mathfrak{U}$.

Model-theoretically, $\mathfrak{U}$ is the Fraisse limit of the Fraisse class of finite metric spaces of diameter ≤ 1; it is the model-completion of the (empty) theory of metric spaces in the signature consisting solely of the metric symbol d.

Key fact

For any $A \subseteq \mathfrak{U}$, $\mathrm{dcl}(A) = \bar{A}$.

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Definable functions in $\mathfrak{U}$

Set-up:

- $f : \mathfrak{U}^n \to \mathfrak{U}$ an A-definable function, where $A \subseteq \mathfrak{U}$
- $\mathbb{U}$ an ω_1-saturated elementary extension of $\mathfrak{U}$
- $\tilde{f} : \mathbb{U}^n \to \mathbb{U}$ the natural extension of f

Theorem (G.-2010)

If $f : \mathfrak{U}^n \to \mathfrak{U}$ is A-definable, then either $\tilde{f}$ is a projection function $(x_1, \ldots, x_n) \mapsto x_i$ or else $\tilde{f}$ has compact image contained in $\bar{A} \subseteq \mathfrak{U}$. Consequently, either f is a projection function or else has relatively compact image.

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Corollaries

Corollary

1. *If* $f : \mathfrak{U} \to \mathfrak{U}$ *is a definable surjective/open/proper map, then* $f = \mathrm{id}_{\mathfrak{U}}$.
2. *If* $f : \mathfrak{U} \to \mathfrak{U}$ *is a definable isometric embedding, then* $f = \mathrm{id}_{\mathfrak{U}}$.
3. *(Ealy, G.) If* $n \geq 2$*, then there are no definable isometric embeddings* $\mathfrak{U}^n \to \mathfrak{U}$.

Reason: Compact sets in $\mathfrak{U}$ have no interior.

Isometric Embeddings $\mathfrak{U} \to \mathfrak{U}$

There are many natural isometric embeddings $\mathfrak{U} \to \mathfrak{U}$, none of which (other than $\mathrm{id}_{\mathfrak{U}}$) are definable in $\mathfrak{U}$.

Examples

1. Suppose that X_1 and X_2 are compact subspaces of $\mathfrak{U}$. Then any isometry $\phi : X_1 \to X_2$ can be extended to an isometry $\tilde{\phi} : \mathfrak{U} \to \mathfrak{U}$.
2. Suppose that $x_1, \ldots, x_n \in \mathfrak{U}$. Define

   $$\mathrm{Med}(x_1, \ldots, x_n) := \{z \in \mathfrak{U} \mid d(z, x_i) = d(z, x_j) \text{ for all } i, j\}.$$

   Then $\mathrm{Med}(x_1, \ldots, x_n)$ is isometric to $\mathfrak{U}$.
3. Suppose that M is a Polish subspace of $\mathfrak{U}$ which is a Heine-Borel subspace. Then for any $R \in (0, 1]$, $\{x \in \mathfrak{U} \mid d(x, M) \geq R\}$ is isometric to $\mathfrak{U}$.

Definable Groups

Corollary

There are no definable group operations on $\mathfrak{U}$.

Cameron and Vershik introduced a group operation on $\mathfrak{U}$ for which there is a dense cyclic subgroup. This group operation allows one to introduce a notion of translation in $\mathfrak{U}$. By the above corollary, this group operation is not definable.

Key Ideas to the Proof for $n = 1$

Suppose that $f : \mathfrak{U} \to \mathfrak{U}$ is an A-definable function, where $A \subseteq \mathfrak{U}$ is countable. Let $\tilde{f} : \mathbb{U} \to \mathbb{U}$ denote its canonical extension.

1. By triviality of dcl, for any $x \in \mathbb{U}$, we have $\tilde{f}(x) \in \mathrm{dcl}(Ax) = \bar{A} \cup \{x\}$.
2. Let $X = \{x \in \mathfrak{U} \mid f(x) = x\}$. Show that $\tilde{f}^{-1}(\bar{A}) \setminus X \subseteq \mathrm{int}(\tilde{f}^{-1}(\bar{A}))$.
3. Prove a general lemma showing that if $F \subseteq \mathbb{U}$ is a closed subset and $G \subseteq F$ is a closed, separable subset of F for which $F \setminus G \subseteq \mathrm{int}(F)$, then either $F = G$ or $F = \mathbb{U}$. This involves a bit of "Urysohn-esque" arguing.
4. Finally, a saturation argument shows that if $\tilde{f}(\mathbb{U}) \subseteq \mathfrak{U}$, then $\tilde{f}(\mathbb{U})$ is compact.

Urysohn-esque arguing

Lemma

Suppose that $F \subseteq \mathbb{U}$ is closed and $G \subseteq F$ is a closed, separable subset of F for which $F \setminus G \subseteq \mathrm{int}(F)$. Then either $F = G$ or $F = \mathbb{U}$.

Proof.

Suppose $F \neq G$. Let $0 < r < d(y, G)$. Cover G with countably many balls of radius r and call the union of these balls B. Set $Y = \mathbb{U} \setminus B$, which is (path-)connected by some Urysohn-esque arguing. Now $F \cap Y = \mathrm{int}(F) \cap Y$ is a nonempty, clopen subset of Y, implying that $F \cap Y = Y$. It follows that $Y \subseteq F$. Since r can be taken to be arbitrarily small, this shows that $\mathbb{U} \setminus G \subseteq F$, whence $F = \mathbb{U}$. □

Question

Question 3

Can we improve the theorem on definable functions to read: If $f : \mathfrak{U}^n \to \mathfrak{U}$ is definable, then either f is a projection or a constant function?

I can show that a positive solution to the above question follows from a positive solution to the $n = 1$ case.

The Case of Relatively Compact Image

In the hopes of answering this question, we can say some things about $\tilde{f}(\mathbb{U}^n)$ in the case that it is relatively compact:

- $\tilde{f}(\mathbb{U}^n)$ is a continuum (connected, compact space).
- Consequently, if $\bar{A}$ is totally disconnected, then $\tilde{f}$ is a constant function.
- $\tilde{f}(\mathbb{U}^n)$ is a perfect space unless it is a singleton.
- If $\tilde{f}(\mathbb{U}^n)$ is not a singleton, then $\tilde{f}(\mathbb{U}^n)$ is either a Peano space (continuous image of $[0, 1]$) or else a reducible continuum (every two points are contained in a proper subcontinuum.)
- Consequently, $\tilde{f}(\mathbb{U}^n)$ is a decomposable continuum. Since the generic continuum is (hereditarily) indecomposable, we see that $\tilde{f}(\mathbb{U}^n)$ is a special kind of continuum.
- $\tilde{f}(\mathbb{U}^n)$ contains *arbitrarily small path-connected subcontinua*.

Question

Question 4

Are there any definable injections $f : \mathfrak{U} \to \mathfrak{U}$ other than the identity?

There can exist injective functions $\mathfrak{U} \to \mathfrak{U}$ which have relatively compact image, so our theorem doesn't immediately help us: Consider

$$(x_n) \mapsto (\frac{x_n}{2^n}) : (0,1)^\infty \to \ell^2.$$

and use the fact that $\mathfrak{U} \cong \ell^2 \cong (0,1)^\infty$.

Observe that a positive answer to Question 3 yields a negative answer to this question.

Injective Definable Functions

Lemma

If $f : \mathbb{U} \to \mathbb{U}$ *is injective and definable, then* $f = \mathrm{id}_{\mathbb{U}}$*.*

Proof.

One can show that the complement of an open ball in $\mathbb{U}$ is definable. Since f maps definable sets to definable sets (which is a fact we are unsure of in $\mathfrak{U}$), it follows that f is a closed map, whence a topological embedding. By our main theorem, we see that f is the identity. □

Remark

This doesn't immediately help us, for an injective definable map $\mathfrak{U} \to \mathfrak{U}$ need not induce an injective definable map $\mathbb{U} \to \mathbb{U}$. (Continuous logic is a positive logic!)

Upwards Transfer

Lemma (BBHU, Ealy-G.)

Suppose that M is ω-satuated and $P, Q : M^n \to [0,1]$ are definable predicates such that P is defined over a finite parameterset. Then the statement " for all $a \in M^n$ $(P(a) = 0 \Rightarrow Q(a) = 0)$" is expressible in continuous logic.

- It follows that the natural extension of an isometric embedding is also an isometric embedding.
- It also follows that if $f : M^n \to M$ is an A-definable injection, where A is *finite*, then $\tilde{f}$ is also an injection.

Hilbert spaces

- Throughout, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.
- Recall that an inner product space over $\mathbb{K}$ which is complete with respect to the metric induced by its inner product is called a $\mathbb{K}$-Hilbert space. In this talk, H and H' denote *infinite-dimensional* $\mathbb{K}$-Hilbert spaces.
- A continuous linear transformation $T : H \to H'$ is also called a *bounded* linear transformation. Reason: if one defines

$$\|T\| := \sup\{\|T(x)\| \ : \ \|x\| \leq 1\},$$

then T is continuous if and only if $\|T\| < \infty$.
- We let $\mathfrak{B}(H)$ denote the (C^*-) algebra of bounded operators on H.

Signature for Real Hilbert Spaces

We work with the following many-sorted metric signature:

- for each $n \geq 1$, we have a sort for $B_n(H) := \{x \in H \mid \|x\| \leq n\}$.
- for each $1 \leq m \leq n$, we have a function symbol $I_{m,n} : B_m(H) \to B_n(H)$ for the inclusion mapping.
- function symbols $+, - : B_n(H) \times B_n(H) \to B_{2n}(H)$;
- function symbols $r\cdot : B_n(H) \to B_{kn}(H)$ for all $r \in \mathbb{R}$, where k is the unique natural number satisfying $k - 1 \leq |r| < k$;
- a predicate symbol $\langle \cdot, \cdot \rangle : B_n(H) \times B_n(H) \to [-n^2, n^2]$;
- a predicate symbol $\| \cdot \| : B_n(H) \to [0, n]$.

The moduli of uniform continuity are the natural ones.

Signature for Complex Hilbert Spaces

When working with complex Hilbert spaces, we make the following changes:

- We add function symbols $i\cdot : B_n(H) \to B_n(H)$ for each $n \geq 1$, meant to be interpreted as multiplication by i.
- Instead of the function symbol $\langle \cdot, \cdot \rangle : B_n(H) \times B_n(H) \to [-n^2, n^2]$, we have two function symbols $\mathfrak{Re}, \mathfrak{Im} : B_n(H) \times B_n(H) \to [-n^2, n^2]$, meant to be interpreted as the real and imaginary parts of $\langle \cdot, \cdot \rangle$.

Definable functions

Definition

Let $A \subseteq H$. We say that a function $f : H \to H$ is *A-definable* if:

(i) for each $n \geq 1$, $f(B_n(H))$ is bounded; in this case, we let $m(n, f) \in \mathbb{N}$ be the minimal m such that $f(B_n(H))$ is contained in $B_m(H)$;

(ii) for each $n \geq 1$ and each $m \geq m(n, f)$, the function

$$f_{n,m} : B_n(H) \to B_m(H), \quad f_{n,m}(x) = f(x)$$

is A-definable, that is, the predicate $P_{n,m} : B_n(H) \times B_m(H) \to [0, m]$ defined by $P_{n,m}(x, y) = d(f(x), y)$ is A-definable.

Lemma

The definable bounded operators on H form a subalgebra of $\mathfrak{B}(H)$.

Statement of the Main Theorem

From now on, $I : H \to H$ denotes the identity operator.

Definition

An operator $K : H \to H$ is *compact* if $K(B_1(H))$ has compact closure. (In terms of nonstandard analysis: K is compact if and only if for all finite vectors $x \in H^*$, we have $K(x)$ is nearstandard.)

Theorem (G.-2010)

The bounded operator $T : H \to H$ is definable if and only if there is $\lambda \in \mathbb{K}$ and a compact operator $K : H \to H$ such that $T = \lambda I + K$. (Definable=scalar + compact)

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Finite-Rank Operators

- Suppose first that T is a *finite-rank* operator, that is, $T(H)$ is finite-dimensional.
- Let $a_1, \ldots, a_n$ be an orthonormal basis for $T(H)$. Then $T(x) = T_1(x)a_1 + \cdots + T_n(x)a_n$ for some bounded linear functionals $T_1, \ldots, T_n : H \to \mathbb{R}$.
- By the Riesz Representation Theorem, there are $b_1, \ldots, b_n \in H$ such that $T_i(x) = \langle x, b_i \rangle$ for all $x \in H$, $i = 1, \ldots, n$.
- Then, for all $x, y \in H$, we have

$$d(T(x), y) = \sqrt{\sum_{i=1}^{n}(\langle x, b_i \rangle^2) - 2\sum_{i=1}^{n}(\langle x, b_i \rangle \langle a_i, y \rangle) + \|y\|^2}$$

which is a formula in our language. Hence, finite-rank operators are strongly definable.

Compact Operators

Fact

If $T : H \to H$ is compact, then there is a sequence (T_n) of finite-rank operators such that $\|T - T_n\| \to 0$ as $n \to \infty$.

- Now suppose that $T : H \to H$ is a compact operator. Fix a sequence (T_n) of finite-rank operators such that $\|T - T_n\| \to 0$.
- Fix $n \geq 1$ and $\epsilon > 0$ and choose k such that $\|T - T_k\| < \frac{\epsilon}{n}$. Then for $x \in B_n(H)$ and $y \in B_m(H)$, where $m \geq m(n, T)$, we have

$$|d(T(x), y) - d(T_k(x), y)| \leq \|T(x) - T_k(x)\| < \epsilon.$$

- Since $d(T_k(x), y)$ is given by a formula, this shows that T is definable.
- Thus, any operator of the form $\lambda I + T$ is definable.

Working towards the converse

- From now on, we fix an A-definable operator $T : H \to H$, where $A \subseteq H$ is countable.
- We also let H^* denote an ω_1-saturated elementary extension of H.
- Observe that, since H is closed in H^*, we have the orthogonal decomposition $H^* = H \oplus H^\perp$.
- T has a natural extension to a definable function $T : H^* \to H^*$.

Lemma

$T : H^* \to H^*$ is also linear.

Proof.

Not as straightforward as you might guess given that continuous logic is a positive logic! □

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Definable closure

Fact

- In a Hilbert space H, $\mathrm{dcl}(B) = \overline{\mathrm{sp}}(B)$, the closed linear span of B, for any $B \subseteq H$.

We let $P : H^* \to H^*$ denote the orthogonal projection onto the subspace $\overline{\mathrm{sp}}(A)$.

Lemma

For any $x \in H^*$, $\mathrm{dcl}(Ax) = \overline{\mathrm{sp}}(Ax) = \overline{\mathrm{sp}}(A) \oplus \mathbb{K} \cdot (x - Px)$.

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Main Lemma

Lemma

There is a unique $\lambda \in \mathbb{K}$ such that, for all $x \in H^*$, we have $T(x) = PT(x) + \lambda(x - Px)$.

Proof.

- If $x \in H^\perp$, then there is $\lambda_x \in \mathbb{K}$ such that $T(x) = PT(x) + \lambda_x \cdot x$.
- It is easy to check that $\lambda_x = \lambda_y$ for all $x, y \in H^\perp$; call this common value λ.
- For $x \in H^*$ arbitrary, we have

$$T(x) = TP(x) + T(x - Px) = TP(x) + PT(x - Px) + \lambda(x - Px).$$

- Since $TP(x) + PT(x - Px) \in \overline{\mathrm{sp}}(A)$, we are done.

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- For $x \in H^*$ arbitrary, we have

$$T(x) = TP(x) + T(x - Px) = TP(x) + PT(x - Px) + \lambda(x - Px).$$

- Since $TP(x) + PT(x - Px) \in \overline{\text{sp}}(A)$, we are done.

Finishing the converse

Proposition

For λ as above, we have $T - \lambda I$ is a compact operator.

Proof

- Since $T - \lambda I = P \circ (T - \lambda I)$, we have $(T - \lambda I)(H^*) \subseteq \overline{\text{sp}}(A)$.
- Let $\epsilon > 0$ be given. Let $\varphi(x, y)$ be a formula such that $\big| \|T(x) - y\| - \varphi(x, y) \big| < \frac{\epsilon}{4}$, where x is a variable of sort B_1.
- Let (b_n) be a countable dense subset of $(T - \lambda I)(B_1(H^*))$.
- Then the following set of statements is inconsistent:

$$\{\|T(x) - (\lambda x + b_n)\| \geq \frac{\epsilon}{4} \mid n \in \mathbb{N}\}.$$

Proof (cont'd)

- Thus, the following set of conditions is inconsistent:

$$\{\varphi(x, \lambda x + b_n) \geq \frac{\epsilon}{2} \mid n \in \mathbb{N}\}.$$

- By ω_1-saturation, there are $b_1, \ldots, b_m$ such that

$$\{\varphi(x, \lambda x + b_n) \geq \frac{\epsilon}{2} \mid 1 \leq n \leq m\}$$

  is inconsistent.
- It follows that $\{b_1, \ldots, b_m\}$ form an ϵ-net for $(T - \lambda I)(B_1(H^*))$.
- Since $\epsilon > 0$ is arbitrary, we see that $(T - \lambda I)(B_1(H^*))$ is totally bounded. It is automatically closed by ω_1-saturation, whence it is compact. □

Some Corollaries- I

Corollary

The definable operators on H form a C^*-subalgebra of $\mathfrak{B}(H)$.

- It is not at all clear how to prove, from first principles, that definable operators are closed under taking adjoints.
- It is easy to show this if one assumes that the definable operator is *normal*, for then one has

$$\begin{aligned}\|T^*(x) - y\|^2 &= \|T^*(x)\|^2 - 2\langle T^*(x), y\rangle + \|y\|^2 \\ &= \|T(x)\|^2 - 2\langle T(y), x\rangle + \|y\|^2.\end{aligned}$$

Some Corollaries-II

Corollary

Suppose that T is definable and not compact. Then $\mathrm{Ker}(T)$ *and* $\mathrm{Coker}(T)$ *are finite-dimensional. Moreover,* $\mathrm{Ker}(T) \subseteq \overline{\mathrm{sp}}(A)$.

Proof.

- The moreover is clear from the main lemma.
- By taking adjoints, it is enough to prove the result for $\mathrm{Ker}(T)$.
- Let $\varphi_k(x, y)$ approximate $d(T(x), y)$ within an error of $\frac{1}{k}$. Then the following set of formulae is inconsistent:

$$\{\varphi_k(x, 0) \leq \frac{1}{k} \ : \ k \geq 1\} \cup \{d(x, a) \geq \epsilon \mid a \in A\}$$

- By ω_1-saturation, there is a finite ϵ-net for $B_1(\mathrm{Ker}(T))$. Thus, $B_1(\mathrm{Ker}(T))$ is compact, whence $\mathrm{Ker}(T)$ is finite-dimensional.

Some Corollaries- III

Corollary

Suppose that E is a closed subspace of H and that $T : H \to H$ is the orthogonal projection onto E. Then T is definable if and only if E has finite dimension or finite codimension.

Corollary

Let $I = \{i_1, i_2, \ldots\}$ be an infinite and coinfinite subset of $\mathbb{N}$. Let $T : \ell^2 \to \ell^2$ be given by $T(x)_n = x_{i_n}$. Then T is not definable.

Fredholm operators

From now on, we assume that $\mathbb{K} = \mathbb{C}$. Recall that a bounded operator T is *Fredholm* if both $\mathrm{Ker}(T)$ and $\mathrm{Coker}(T)$ are finite-dimensional. The *index* of a Fredholm operator is the number $\mathrm{index}(T) := \dim(\mathrm{Ker}(T)) - \dim(\mathrm{Coker}(T))$.

Corollary

If T is definable, then either T is compact or else T is Fredholm of index 0.

Proof.

This follows from the Fredholm alternative of functional analysis. □

Some Corollaries- IV

Recall the left- and right-shift operators L and R on ℓ^2:

$$L(x_1, x_2, \ldots,) = (x_2, x_3, \ldots)$$

$$R(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots,)$$

Corollary

The left- and right-shift operators on ℓ^2 are not definable.

Proof.

These operators are of index 1 and -1 respectively. □

Using this result, one can prove that the left-and right-shift operators on the $\mathbb{R}$-Hilbert space ℓ^2 are not definable.

The Calkin Algebra

- Let $\mathfrak{B}_0(H)$ denote the ideal of $\mathfrak{B}(H)$ consisting of the compact operators. The quotient algebra $\mathfrak{C}(H) = \mathfrak{B}(H)/\mathfrak{B}_0(H)$ is referred to as the *Calkin algebra* of H.
- Let $\pi : \mathfrak{B}(H) \to \mathfrak{C}(H)$ be the canonical quotient map.
- Our main theorem says that the algebra of definable operators is equal to $\pi^{-1}(\mathbb{C})$.
- We consider the *essential spectrum* of T:

$$\sigma_e(T) = \{\lambda \in \mathbb{C} \ : \ \pi(T) - \lambda \cdot \pi(I) \text{ is not invertible}\}.$$

Some Corollaries- V

If T is a definable operator, let $\lambda(T) \in \mathbb{C}$ be such that $T - \lambda(T)I = P \circ (T - \lambda(T)I)$.

Corollary

If T is definable, then $\sigma_e(T) = \{\lambda(T)\}$.

Example

Consider $L \oplus R : \ell^2 \oplus \ell^2 \to \ell^2 \oplus \ell^2$.

- It is a fact that $L \oplus R$ is Fredholm of index 0. Thus, our earlier corollary doesn't help us in showing that $L \oplus R$ is not definable.
- However, it is a fact that $\sigma_e(L \oplus R) = \mathbb{S}^1$. Thus, we see from the above corollary that $L \oplus R$ is not definable.

The Invariant Subspace Problem

Invariant Subspace Problem

If H is a separable Hilbert space and $T : H \to H$ is a bounded operator, does there exist a closed subspace E of H such that $E \neq \{0\}$, $E \neq H$, and $T(E) \subseteq E$?

Silly Corollary

The invariant subspace problem has a positive answer when restricted to the class of *definable* operators.

Proof.

Suppose T is definable. Write $T = \lambda I + K$. If $K = 0$, then $E := \mathbb{C} \cdot x$ is a closed, nontrivial invariant subspace for T, where $x \in H \setminus \{0\}$ is arbitrary. Otherwise, use the fact that compact operators always have nontrivial invariant subspaces. □

Herbrand's Theorem

Theorem (Classical Logic)

Suppose that T is a universal $\mathcal{L}$-theory that admits quantifier elimination. Let $\vec{x} = (x_1, \ldots, x_m)$ and $\vec{y} = (y_1, \ldots, y_n)$. Then for any formula $\varphi(\vec{x}, \vec{y})$, there are $\mathcal{L}$-terms

$$t_{11}(\vec{x}), \ldots, t_{k1}(\vec{x}), \ldots, , t_{k1}(\vec{x}), \ldots, t_{kn}(\vec{x})$$

such that, for any $M \models T$ and any $\vec{a} \in M^m$, if $M \models \exists \vec{y} \varphi(\vec{a}, \vec{y})$, then

$$M \models \bigvee_{1 \leq i \leq k} \varphi(\vec{a}, t_{i1}(\vec{a}), \ldots, t_{in}(\vec{a})).$$

An Approximate Herbrand's Theorem

Theorem (G.-2011)

Suppose that T is a universal $\mathcal{L}$-theory that admits quantifier elimination. Let $\vec{x} = (x_1, \ldots, x_m)$ and $\vec{y} = (y_1, \ldots, y_n)$. Then for any formula $\varphi(\vec{x}, \vec{y})$ and any $\epsilon > 0$, there are $\mathcal{L}$-terms

$$t_{11}(\vec{x}), \ldots, t_{k1}(\vec{x}), \ldots, , t_{k1}(\vec{x}), \ldots, t_{kn}(\vec{x})$$

such that, for any $M \models T$ and any $\vec{a} \in M^m$, if $M \models (\inf_{\vec{y}} \varphi(\vec{a}, \vec{y})) = 0$, then

$$M \models \min_{1 \leq i \leq k} \varphi(\vec{a}, t_{i1}(\vec{a}), \ldots, t_{in}(\vec{a})) \leq \epsilon.$$

Approximations to Definable Functions

Lemma

If T is a model-complete, $\exists\forall$-theory and $M \models T$, then $\mathrm{Th}(M_M)$ is universal and admits quantifier elimination.

Corollary

Suppose that T is a model-complete, $\exists\forall$-theory in the signature $\mathcal{L}$. Suppose that $M \models T$ and $f : M^n \to M$ is a definable function. Then for any $\epsilon > 0$, there are $\mathcal{L}(M)$-terms $t_1(\vec{x}), \ldots, t_k(\vec{x})$ such that: for all $\vec{a} \in M^n$, there is $i \in \{1, \ldots, k\}$ with $d(f(\vec{a}), t_i(\vec{a})) \leq \epsilon$.

Hilbert spaces and generic expansions

The following are (complete) model-complete $\exists\forall$-theories:

- (infinite-dimensional) Hilbert spaces
- Hilbert spaces expanded by a generic automorphism
- Hilbert spaces expanded by a generic unitary representation
- Hilbert spaces expanded by a generic subspace (beautiful pairs of Hilbert spaces)

Thus, in models of these theories, definable functions are approximately piecewise given by terms.

Remark

These theories actually have QE, so we could just analyze definable closure to obtain the above result for these structures as well. This is a bit messier.

Other Metric Structures?

Question

What about definable functions in other metric structures? E.g. atomless probability algebras? These are $\forall\exists$, so the approximate Herbrand doesn't apply.

Definable sets

A closed set $X \subseteq M^m$ is *A-definable* if the predicate $x \mapsto d(x, X) : M^m \to [0, 1]$ is A-definable.

Question

What about definable sets in metric structures?

This last question seems harder. For example, any compact subset of any metric structure is definable. Therefore any compact metric space (of diameter ≤ 1) is a definable set in $\mathfrak{U}$.

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Musings on Definable Sets in $\mathfrak{U}$

By the strong ω-categoricity of $T_{\mathfrak{U}}$, we have that, for finite $A \subseteq \mathfrak{U}$ and closed $X \subseteq \mathfrak{U}^m$, the following are equivalent:

- X is A-definable;
- X is A-type-definable;
- X is an A-zeroset;
- X is invariant under $\mathrm{Aut}(\mathfrak{U}/A)$.

Musing on Definable Sets in $\mathfrak{U}$ (cont'd)

Consequently, for A-definable $X, Y \subseteq \mathfrak{U}$, we have:

- ∂X, $\overline{\text{int}(X)}$, $\overline{\mathfrak{U} \setminus X}$, $X \cap Y$, and $\text{Ker}(X)$ are A-definable.
- If X is connected, then X is a "generalized annulus". This gives, for any $a \in \mathfrak{U}$, $2^{\aleph_0}$ many non-isometric $\{a\}$-definable sets.
- The connected components of X are A-definable and any 1-element connected subset of X must be an element of A. Moreover, if there are infinitely many connected components of X, then they cannot be a uniform distance apart.
- If X is compact, then X is a (finite) subset of A.

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Preprints for these papers are available at

www.math.ucla.edu/ $\sim$ isaac