Group Colorings and Bernoulli Subflows

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This is joint work with Steve Jackson and Brandon Seward.

A coloring property for countable groups, Mathematical Proceedings of the Cambridge Philosophical Society 147 (2009), no. 3, 579–592.

Group colorings and Bernoulli subflows, manuscript in preparation.

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The answer was known to be yes for $\mathbb{Z}$, $S_{<\infty}$, torsion-free hyperbolic groups (including the free groups), residually finite groups, etc. (Dranishnikov-Shroeder, Glasner-Uspenskij)

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Theorem (GJS, 2008)
For every countably infinite group G there exists a free Bernoulli subflow.

Constructing free subflows

$\Longleftrightarrow$

constructing $x \in 2^G$ so that $\overline{[x]} \subseteq F(G)$
i.e., $x \in 2^G$ such that every $y \in \overline{[x]}$ is aperiodic

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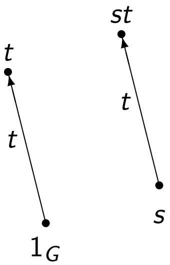

2-Colorings

Let G be a countable group. A 2-coloring on G is a function $x : G \to \{0,1\}$ such that

> *for any $s \in G$ with $s \neq 1_G$, there is a finite set $T \subseteq G$ such that*

$$\forall g \in G\ \exists t \in T\ x(gt) \neq x(gst).$$

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Lemma (GJS, Pestov)
x is a 2-coloring on G iff $\overline{[x]}$ is a free subflow.

for any $s \in G$ with $s \neq 1_G$, there is a finite set $T \subseteq G$ such that

$$\forall g \in G \; \exists t \in T \; x(gt) \neq x(gst).$$

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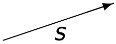

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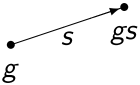

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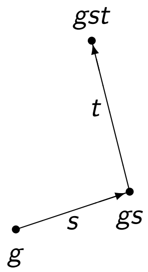

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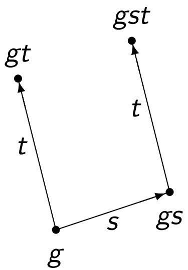

On $\mathbb{Z}$ it is fairly easy to construct aperiodic elements, and it is significantly harder to construct 2-colorings.

In particular, any 2-coloring cannot contain arbitrarily long subsequences of 1's; otherwise the constant 1 element (certainly periodic!) would be a limit point of the orbit.

Since 2-colorings (especially on general countable groups) are not easy to construct, we certainly hope that it is then not easy to destroy the 2-coloring property!

Question If x is a 2-coloring on G and $y =^* x$ (i.e. $\{g \in G : x(g) \neq y(g)\}$ is finite), is y necessarily a 2-coloring?

Strong 2-colorings

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Corollary

For any countably infinite group G the set of all 2-colorings on G is dense.

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Lemma

x is a strong 2-coloring iff x is a 2-coloring and for any $1_G \neq s \in G$ there are *infinitely many* $t \in G$ such that $x(t) \neq x(st)$.

Almost 2-colorings

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G has the ACP iff every 2-coloring on G is a strong 2-coloring iff every almost 2-coloring on G is a strong 2-coloring.

Near 2-colorings

Let G be a countable group, $x \in 2^G$, and $1_G \neq s \in G$. We say that x nearly blocks s if there are finite sets $S, T \subseteq G$ such that

$$\forall g \notin S\ \exists t \in T\ x(gt) \neq x(gst).$$

x is a near 2-coloring if x nearly blocks s for all $1_G \neq s \in G$.

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Obviously

strong 2-coloring $\Longrightarrow$ *2-coloring* $\Longrightarrow$ *almost 2-coloring* $\Longrightarrow$ *near 2-coloring*

Lemma

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Call $x \in 2^G$ a pathological periodic element if it is a periodic almost 2-coloring.

ACP $\Leftrightarrow$ there is no pathological periodic element in 2^G.

ACP

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Lemma

Let G be countably infinite and x an almost 2-coloring on G. Then the stabilizer of x

$$\{g \in G : g \cdot x = x\}$$

is finite.

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We also showed that ACP holds for nilpotent groups, FC groups, etc.

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Fact: $\mathbb{Z}_2 * \mathbb{Z}_2$ is almost abelian (solvable of rank 2).

$$\begin{array}{ccccccc} 1 & a & ab & aba & (ab)^2 & (ab)^2a & \cdots \\ b & ba & bab & (ba)^2 & b(ab)^2 & (ba)^3 & \cdots \end{array}$$

A Complete Characterization of ACP

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Theorem

Let G be a countably infinite group. Then G has the ACP iff for any $g \in G$, there is $h \in \langle g \rangle$ such that the centralizer of h

$$C(h) = \{k \in G : kh = hk\}$$

is infinite.

Almost Equality and Indestructibility of Periodicity

Theorem TFAE for a countably infinite group G:
(1) There is an "indestructible" periodic element, i.e., there is a periodic $x \in 2^G$ such that any $y =^* x$ is also periodic.
(2) G contains a nonabelian free subgroup.

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Theorem TFAE for a countably infinite group G:
(1) There is a "not easily destructible" periodic element, i.e., there is a periodic $x \in 2^G$ such that every $y =^{**} x$ is also periodic.
(2) G contains a subgroup which is a free product of nontrivial groups.

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Fact: There is no minimal pathological periodic element. (If $x =^* y$ then $\overline{[x]} - [x] \subseteq \overline{[y]}$.)

Thank you!