Martin-Löf randomness, invariant measures and countable homogeneous structures II

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1 Introduction

The focus of this work is on the problem of understanding the symmetries that transform a recursively presented universal structure,

which in this paper will be a Fraïssé limit of finite first order structures,

to a copy of such a structure which is Martin-Löf random relative to an S_∞-invariant measure on the class of all universal structures of the given type.

Here S_∞ is the symmetric group of a countable set, with the pointwise convergence topology.

This invesigation leads to a link between the so-called discernable flows in structural Ramsey theory and algorithmic randomness.

2 Preliminaries on amenable groups

Let G be a topological group and X a compact Hausdorff space. A dynamical system (X, G) (or a G-flow on X) is given by a jointly continuous action of G on X. If (Y, G) is a second dynamical system, then a G-morphism $\pi : (X, G) \to (Y, G)$ is a continuous mapping $\pi : X \to Y$ which intertwines the G-actions, i.e.,the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\alpha} & X \\ {\scriptstyle 1\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times Y & \xrightarrow[\beta]{} & Y \end{array},$$

commutes with α, β being the group actions.

An isomorphism is a bijective homomorphism. A subflow of (X, G) is a G-flow on a compact subspace Y of X with the action the restricted to the action of G on X to the action on Y.

A G-flow is minimal if it has no proper subflows. Every dynamical system has a minimal subflow (Zorn).

The following fact, first proven by Ellis (1949) [5], is central to the theory of dynamical systems:

Theorem 1 *Let G be a Hausdorff topological group. There exists, up to G-isomorphism, a unique minimal dynamical system, denoted by $(M(G), G)$, such that for every minimal dynamical system (X, G) there exists a G–epimorphism*

$$\pi : (M, G) \longrightarrow (X, G),$$

and any two such universal systems are isomorphic.

The flow $(M(G), G)$ is called the *universal minimal flow* of G.

We next introduce the notion of amenable groups.

Definition 1 *A topological group G is amenable if, whenever X is a non-empty compact Hausdorff space and π is a continuous action of G on X, then there is a G–invariant Borel probability measure on X.*

3 Some structural Ramsey theory

Let $\mathbf{K}$ be the age of some countable $\mathcal{L}$-structure. For $A, \pi \in \mathbf{K}$ we denote by A^π the set of all the (model-theoretic) structure-preserving embeddings of π in A. For a natural number $r \geq 1$ and for $\pi, A, B \in \mathbf{K}$ we introduce the predicate $B \rightsquigarrow (A)^\pi_r$ (Erdős-notation) to mean:

$$B \rightsquigarrow (A)^\pi_r := \begin{array}{ccc} A & \cdots\cdots & B \\ & \diagdown \quad \diagup_{r} & \\ & \pi & \end{array} \iff \Big(\forall \; B^\pi \xrightarrow{\chi} r \;\; \exists \; A \rightarrowtail^{\alpha} B \quad \begin{array}{ccc} A^\pi & \xrightarrow{\alpha_*} & B^\pi \\ {}_{!}\searrow & \oplus & \swarrow_{\chi} \\ & r & \end{array} \Big).$$

Here $\alpha_* : A^\pi \to B^\pi$ is the mapping that takes an embedding $\pi \overset{x}{\rightarrowtail} A$ to the induced embedding $\pi \overset{\alpha x}{\rightarrowtail} B$.

In other words, $B \rightsquigarrow (A)^\pi_r$ iff: for every r-colouring χ of the set B^π consisting of the embeddings of π in B (copies of π in B), there is an embedding α of A into B such that $\chi\alpha_*$ is a constant. This means that χ assumes a constant value on all the embeddings of π into the image $A' \subset B$ of A under α.

We shall call an age $\mathbf{K}$ a *Ramsey age* if, for all $\pi, A \in \mathbf{K}$ with $A^\pi \neq \emptyset$, and all natural numbers $r \geq 1$, there is some $B \in \mathbf{K}$ such that $B \rightsquigarrow (A)^\pi_r$.

Assume $\mathcal{L}$ is a countable signature containing a distinguished binary relation symbol $<$.

An *order structure* A for the signature $\mathcal{L}$ with the distinguished symbol $<$, is a structure A for which the interpretation $<^A$ of the symbol $<$ in A is a total ordering.

An *order class* $\mathbf{K}$ for $\mathcal{L}$ is one for which all $A \in \mathbf{K}$ are order structures (relative to the distinguished $<$).

Let $\mathcal{L}_0$ be the signature obtained by removing the distinguished symbol $<$ from $\mathcal{L}$. For any $\mathcal{L}$-structure A, denote by A_0 the $\mathcal{L}_0$-structure which is the reduct of A to $\mathcal{L}_0$. This means that A_0 is the structure A where the distinguished order $<$ interpretated as a total order $<^A$ in A is being ignored.

Let $\mathbf{K}$ be a Fraïssé order class. Denote by $\mathbf{K}_0$ the class of all reducts A_0 for $A \in \mathbf{K}$. Write $\mathbb{F}$ for the Fraïssé limit of $\mathbf{K}$. We now discuss when $\mathbf{K}_0$ is also a Fraïssé class with limit $\mathbb{F}_0$ the latter being the reduct of $\mathbb{F}$ to $\mathcal{L}_0$.

We say that the class $\mathbf{K}$ is *reasonable* if for every $A_0, B_0 \in \mathbf{K}_0$, and linear ordering $<$ on A_0 such that $(A_0, <) \in \mathbf{K}$, and for an embedding $\pi : A_0 \to B_0$, there is a linear ordering $<_1$ on B_0 so that $B = (B_0, <_1,) \in \mathbf{K}$ and $\pi : A \to B$ is also an embedding. (This means that $x < y \Leftrightarrow \pi(x) <_1 \pi(y)$.)

Then Kechris et al (p 135) showed that $\mathbf{K}_0$ is a Fraïssé class with limit $\mathbb{F}_0$, (which is the reduct of $\mathbb{F}$ to the signature $\mathcal{L}_0$) iff the Fraïssé order class $\mathbf{K}$ is reasonable.

Note that, in this case, the underlying sets of $\mathbb{F}$ and $\mathbb{F}_0$ are the same. Moreover, we can write

$$\mathbb{F} = (\mathbb{F}_0, <_0),$$

for some linear ordering $<_0$ on the underlying set of $\mathbb{F}_0$.

We consider the continuous action of the automorphism group $\mathrm{Aut}(\mathbb{F}_0)$ on the (topological) space of all linear orderings on the set F_0, which is the underlying set of the structure $\mathbb{F}_0$. Write $X_{\mathbf{K}} \subset \{0,1\}^{F_0 \times F_0}$ for the orbit topological closure of the action of $\mathrm{Aut}(\mathbb{F}_0)$ on the linear ordering $<_0$, i.e.,

$$X_{\mathbf{K}} = \overline{\mathrm{Aut}(\mathbb{F}_0). <_0}.$$

This set is clearly a closed, hence compact, subset of the Baire space $\{0,1\}^{F_0 \times F_0}$. Moreover, it is clearly also an $\mathrm{Aut}(\mathbb{F}_0)$-invariant subset of $\{0,1\}^{F_0 \times F_0}$ under the natural action of $\mathrm{Aut}(\mathbb{F}_0)$ on the latter space. We have thus obtained an $\mathrm{Aut}(\mathbb{F}_0)$-flow on $X_{\mathbf{K}}$.

This flow can be defined for any reasonable (in the technical sense as explained above) Fraïssé order class $\mathbf{K}$. I will call it the *discerning* flow associated with the reasonable Fraïssé order class $\mathbf{K}$ and denote it by $\mathcal{D}(\mathbf{K})$.

If, in addition to being a Fraïssé order class, the class **K** is Ramsey, then every minimal subflow of the discernable flow is isomorphic to the universal minimal flow of $\mathrm{Aut}(\mathbb{F}_0)$.

The Fraïssé order class $\mathbf{K}$ is said to have the *ordering property* if for every $A_0 \in \mathbf{K}_0$, there is a $B_0 \in \mathbf{K}_0$ such for any linear ordering $<$ on A_0 and every linear ordering $<_1$ on B_0, where both $<, <_1$ are restrictions of $<_0$, there is an embedding of $(A_0, <)$ into $(B_0, <_1)$.

The discerning flow associated with Fraïssé order class $\mathbf{K}$ is itself minimal iff $\mathbf{K}$ has the ordering property.

Hence the discerning flow on $X_{\mathbf{K}}$ is the universal minimal $\mathrm{Aut}(\mathbb{F}_0)$-flow iff the Fraïssé order class $\mathbf{K}$ satisfies both the Ramsey and ordering properties.

Example

It is known (F 1997) that the class **P** (finite posets, linear extensions) is Ramsey and has the ordering property. In this case the discerning $\mathrm{Aut}(\mathbb{P}_0)$-flow is thus a universal minimal flow. It acts on the space $X_{\mathbf{P}}$ consisting of the linear extensions of the universal poset $\mathbb{P}_0$.

Using these facts, Kechris and Sokič (2011) showed that the automorphism group of $\mathbb{P}_0$ is not amenable.

These results do imply that the set of linear extensions of the Fraïssé limit of finite posets are all, in a definite sense, nonrandom, at least from the point of view of algorithmic randomness.......

4 Martin-Löf random countable orders

Let S_∞ be the group of permutations of a countable set, which, without loss of generality, we may take to be $\mathbb{N}$. We place on S_∞ the pointwise convergence topolopy.

Let $(\mathbb{N} \times \mathbb{N})_{\neq}$ denote the set of ordered pairs (i, j) of natural numbers with $i \neq j$. Write $\mathcal{M}$ for the set of total orders on $\mathbb{N}$.

We identify $\mathcal{M}$ with a subset of $\{0, 1\}^{(\mathbb{N}\times\mathbb{N})_{\neq}}$ by identifying a total order $<$ on $\mathbb{N}$ with the function $\xi : (\mathbb{N} \times \mathbb{N})_{\neq} \to \{0, 1\}$ given by

$$\xi(x, y) = 1 \Leftrightarrow x < y, \quad x, y \in \mathbb{N}.$$

The total order associated with ξ will be denoted by $<_\xi$. We topologise $\mathcal{M}$ via the natural injection

$$\mathcal{M} \longrightarrow \{0,1\}^{(\mathbb{N}\times\mathbb{N})_{\neq}},$$

where the (Baire) space $\{0,1\}^{(\mathbb{N}\times\mathbb{N})_{\neq}}$ has the product topology.

As such $\mathcal{M}$ is a closed hence compact subspace of $\{0,1\}^{(\mathbb{N}\times\mathbb{N})_{\neq}}$.

The group S_∞ acts continuously on $\mathcal{M}$ if, for $\xi \in \mathcal{M}$ and $\sigma \in S_\infty$, we define the total order $\sigma\xi$ by:

$$x <_{\sigma\xi} y \Longleftrightarrow \sigma^{-1}x <_{\xi} \sigma^{-1}y, \; x, y \in \mathbb{N}.$$

Since S_∞ is an amenable group, there is an S_∞-invariant measure on $\mathcal{M}$.

In fact, Glasner and Weiss (2002) [9] showed that there is *exactly one* such measure (i.e., the flow on $\mathcal{M}$ is uniquely ergodic). Their proof is based on an ergodic argument.

Let us denote this measure by μ. I shall refer to this measure as the *Glasner-Weiss measure.*

In this work, we wish to understand the μ-Martin-Löf random elements of $\mathcal{M}$.

This is a viable project, for it can be shown, with the benefit of hindsight, that the Glasner-Weiss measure can be computed and effectively constructed. (F 2011)

Write $ML_\mu \subset \mathcal{M}$ for the set of μ-Martin-Löf random total orders. Note that $\mu(ML_\mu) = 1$.

Theorem 2 *(F 2011) Let* $\mathbf{K}$ *be a recursive Fraïssé order class which is Ramsey and has the ordering property. Write*

$$\mathbb{F} = (\mathbb{F}_0, <)$$

for its Fraïssé limit and $X_{\mathbf{K}}$ *for the associated discerning flow. Fix some recursive representation of* $\mathbb{F}$*. Note that*

$$X_{\mathbf{K}} \subset \mathcal{M}.$$

If some element of $X_{\mathbf{K}}$ *is* μ*-Martin-Löf random, then the automorphism group* $Aut(\mathbb{F}_0)$ *is amenable. Equivalently, if* $Aut(\mathbb{F}_0)$ *is not amenable, then*

$$ML_\mu \cap X_{\mathbf{K}} = \emptyset.$$

Moreover, if for some computable $\xi \in X_{\mathbf{K}}$ *and some automorphism* π *of* $\mathbb{F}_0$ *it is the case that the linear order* $\pi\xi$ *is* μ*-Martin-Löf random, then* $Aut(\mathbb{F}_0)$ *is amenable.*

Corollary 1 *Fix a recursive representation of the universal poset* $\mathbb{P}$ *on the natural numbers* $\mathbb{N}$*. Let* $\mathcal{M}(\mathbb{P})$ *be the class of linear extensions of* $\mathbb{P}$*. Write* ML_μ *for the set of total orders on* $\mathbb{N}$ *that are Martin-Löf random relative to the Glasner-Weiss probability measure* μ*. Then*

$$ML_\mu \cap \mathcal{M}(\mathbb{P}) = \emptyset.$$

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