Computably enumerable partial orders

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Introduction

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Partial orders, in particular, have been investigated extensively.

Downey, Hirschfeldt, Lempp, and Solomon (2003) – Szpilrajn's Theorem

Hirschfeldt and Shore (2007) – CAC and ADS

Jockusch, Kjos-Hanssen, Lempp, Lerman, and Solomon (2009) – Notions of stability for partial orders

Greenberg, Montalbán, and Slaman (2011) – Degree spectra of linear orders

Introduction

There are several approaches taken in such analyses.

> Most commonly, we restrict attention to computable orders and study the effectivity (or lack thereof) of particular combinatorial constructions or objects of interest.
>
> In computable model theory we might consider noncomputable orders, and inquire instead about which ones admit computable (isomorphic) copies, or more generally, in which Turing degrees copies can be found and how complicated the witnessing isomorphisms are.
>
> In reverse mathematics we formalize theorems pertaining to partial orders, and calibrate the strengths of these theorems according to which set-existence axioms are necessary to prove them.

There is a fruitful interplay between these approaches.

Computably enumerable partial orders

We restrict to partial orders on ω, and identify these with their relations.

Thus, a partial order $(\omega, \leq_P)$ is c.e. if $\leq_P$ is a c.e. binary relation. Similarly for co-c.e. partial orders.

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Theorem (Roy, 1993). There exists a c.e. antisymmetric binary relation which is not isomorphic to any computable such relation.

Theorem (Cholak, Dzhafarov, Schweber, Shore). There exists a co-c.e. partial order on ω which is not isomorphic to any c.e. such partial order, and conversely.

Proof. We first code numbers as follows.

Fix markers a, b, c, f and l.

Partition the rest of ω as follows, with the following relations under $\leq_P$:

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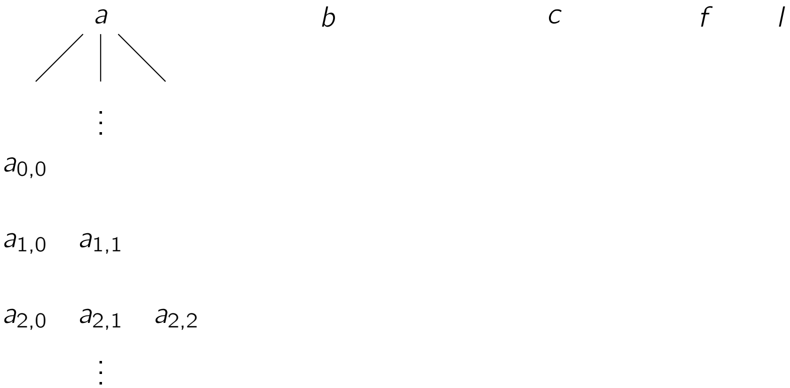

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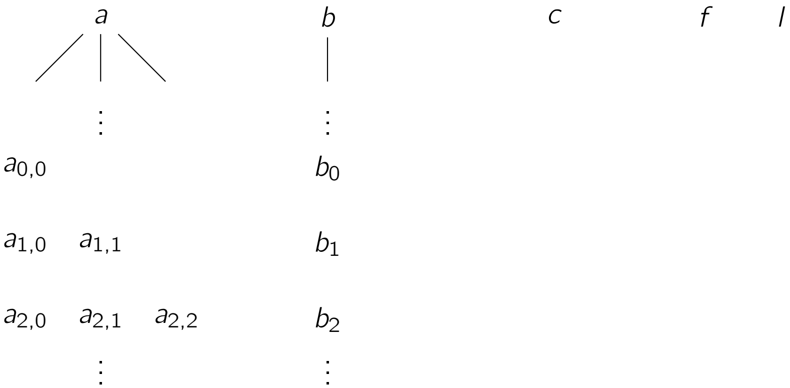

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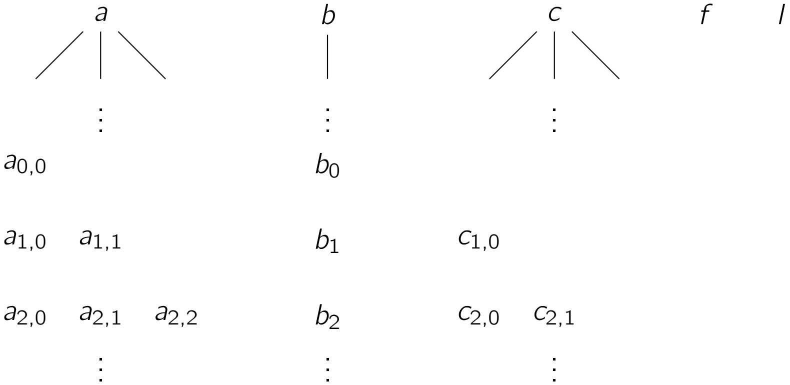
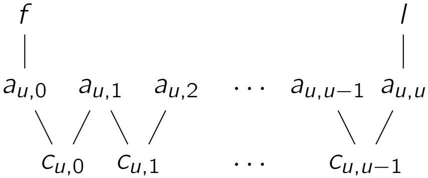


Computably enumerable partial orders

We ensure the coding is recoverable computably in any copy.

The number u is coded by:



Computably enumerable partial orders

Let U be a Σ^0_2-complete set, with R computable such that $u \in U \iff (\exists x)(\forall y) R(u, x, y)$.

Initially, set $b_x \leq a_{u,k}$ for all u and k.

For each u and x, if there is a y such that $\neg R(u, x, y)$, set $b_x \not\leq_P a_{u,k}$ for all $k < u + 1$.

Then $u \in U$ if and only if there is an x such that $b_x \leq_P a_{u,0}, \ldots, a_{u,u}$.

If $\leq_P$ had a c.e. copy, U would then be Σ^0_1-definable, contradiction. $\square$

Generalizations of ADS and CAC

A class of relations related to co-c.e. partial orders is that of inclusion orders on families of sets. For computable families, inclusion is co-c.e.

But since a family of sets $\langle A_i : i \in \omega \rangle$ may have repetitions, we do not necessarily obtain a co-c.e. partial order isomorphic to the inclusion order on simply by setting $i \leq j$ if $A_i \subseteq A_j$. However, we do obviously obtain a preorder.

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This is a characterization:

Proposition. Every co-c.e. preorder on ω is isomorphic to the inclusion order on a computable family of sets.

Generalizations of ADS and CAC

Recall the following combinatorial principles:

Chain/antichain principle (CAC). Every partial order on $\mathbb{N}$ has either an infinite chain or an infinite antichain.

Ascending/descending sequence principle (ADS). Every linear order on $\mathbb{N}$ has either an infinite ascending sequence or an infinite descending sequence.

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These principles were studied in the context of computability theory and reverse mathematics by Hirschfeldt and Shore (2007).

It is easy to see that CAC implies ADS over RCA_0. The converse is open.

Generalizations of ADS and CAC

We can formalize the notion of a c.e. order on $\mathbb{N}$ in RCA_0. Formally, this is a function whose range consists of pairs that satisfy the axioms of a partial order.

We can similarly formalize co-c.e. partial orders, as well as c.e. preorders and co-c.e. preorders.

Thus, we can formulate analogues of CAC and ADS for c.e. and co-c.e. partial orders, and c.e. and co-c.e. preorders, and study their proof-theoretic strength.

Generalizations of ADS and CAC

Proposition (Cholak, Dzhafarov, Schweber, and Shore). Over RCA_0, the following are equivalent:

(1) ADS;
(2) ADS for c.e. partial orders on $\mathbb{N}$;
(3) ADS for co-c.e. partial orders on $\mathbb{N}$;
(4) ADS for c.e. preorders on $\mathbb{N}$;
(5) ADS for co-c.e. preorders on $\mathbb{N}$;
(6) ADS for preorders on $\mathbb{N}$.

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Proof sketch. Straightforward, except for the implications (1) $\implies$ (4) and (1) $\implies$ (5). Here a use of $B\Sigma^0_2$ is necessary, in the form

If $\langle A_i : i \in \mathbb{N}\rangle$ is a family of sets such that for some finite $F \neq \emptyset$, each A_i equals some A_j with $j \in F$, then $\{A_i : A_i = A_j\}$ is infinite for some $j \in F$. □

Generalizations of ADS and CAC

Theorem (Cholak, Dzhafarov, Schweber, and Shore).

(1) There exists a co-c.e. partial order on ω with no infinite antichains and with all chains computing $\emptyset'$.
(2) There exists a c.e. partial order on ω with no infinite chains and with all infinite antichains computing $\emptyset'$.

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Corollary Over RCA_0, the following are equivalent:

(1) ACA_0;
(2) CAC for c.e. partial orders on $\mathbb{N}$;
(3) CAC for co-c.e. partial orders on $\mathbb{N}$;
(4) CAC for c.e. preorders on $\mathbb{N}$;
(5) CAC for co-c.e. preorders on $\mathbb{N}$.

Generalizations of ADS and CAC

Proof of theorem. Let t_i be the least s such that $\emptyset'_s \upharpoonright i = \emptyset' \upharpoonright i$ (in some fixed enumeration)

We build a co-c.e. partial order $\leq_P$ by stages. Initially, make $\leq_P$ agree with the natural order on ω.

At stage $s > 0$, we consider consecutive substages $i \leq s$.

> At substage i, if no number enters $\emptyset' \upharpoonright i$ at stage s, we do nothing and go either to substage $i+1$ or to stage $s+1$, depending on whether $i < s$ or $i = s$.
>
> Otherwise, for all j, k with $i \leq j < k \leq s$, we make $j \nleq_P k$ and go to stage $s+1$.

Clearly, there are no anti chains, and for any chain $c_0 \leq_P c_1 \leq_P \cdots$ we have $t_i \leq c_{i+1}$.

Universality of degree spectra

Recall that the degree spectrum of a countable structure $\mathcal{S}$ is the set of Turing degrees of copies of $\mathcal{S}$.

The study of degree spectra, and in particular, of which classes of degrees can be realized as spectra, has been the subject of many investigations in computable model theory.

Every structure $\mathcal{S}$ we consider below will be assumed to be in a computable language, with computable signature, and automorphically nontrivial. By Knight's theorem, the degree spectrum of any such $\mathcal{S}$ is closed upwards.

Universality of degree spectra

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In fact, more is true:

Theorem (Cholak, Dzhafarov, Schweber, and Shore). For every $\emptyset'$-computable structure $\mathcal{S}$ on ω there exists a c.e. (and a co-c.e.) partial order on ω with the same degree spectrum as $\mathcal{S}$. Furthermore, there exists such a partial order in every c.e. degree containing a copy of $\mathcal{S}$.

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So the degree spectra of c.e. partial orders are universal for $\emptyset'$-computable structures.

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For a nice consequence of the theorem, recall the following well-known result:

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Corollary (Cholak, Dzhafarov, Schweber, and Shore). Every nonzero c.e. degree contains a c.e. (and a co-c.e.) partial order on ω whose degree spectrum consists precisely of the nonzero degrees.

Proof. Fix any structure $\mathcal{S}$ satisfying the Slaman-Wehner theorem, and let $\mathbf{d} > \mathbf{0}$ be c.e. Then $\mathcal{S}$ has a copy in $\mathbf{d}$, and this copy must have the same degree spectrum as $\mathcal{S}$. By the theorem, there is a c.e. partial order $\leq_P$ in $\mathbf{d}$ with the same degree spectrum as $\mathcal{S}$.

Proof of the theorem

Lemma 1 (Folklore). For every structure $\mathcal{S}$ with domain ω there exists a graph R on ω such that $R \equiv_T \mathcal{S}$ and R and $\mathcal{S}$ have the same degree spectrum.

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Lemma 2 (Cholak, Dzhafarov, Schweber, and Shore). For every $\emptyset'$-computable graph R on ω there exists a c.e. (and a co-c.e.) partial order $\leq_P$ on ω with the same degree spectrum. Furthermore, if R has c.e. degree then $\deg(\leq_P) = \deg(R)$.

Proof of the theorem

Proof of Lemma 2. Let a $\emptyset'$-computable graph R be given.

Fix markers a, g, r_0, and r_1.

Partition the rest of ω as follows, with the following relations under $\leq_P$:

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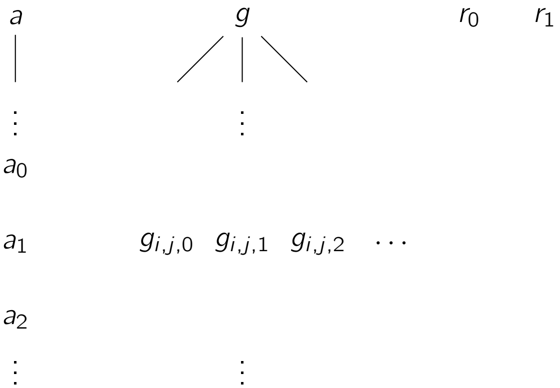

Proof of the theorem

Intuitively:

The a_i represent the elements of the domain of a given copy of R.

The $g_{i,j,k}$ represent guesses at whether or not R holds of the pair (a_i, a_j) in that copy.

r_0 and r_1 code these guesses, with r_0 coding that $R(a_i, a_j)$ does not hold, and r_1 that it does.

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So, initially, we set $g_{i,j,k} \leq_P a_i, a_j$ for all $i < j$ and all k, and let no additional relations hold.

Proof of the theorem

Let R_s be a computable approximation to R such that $\lim_s R_s(i,j) = R(i,j)$ for all $i < j$.

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Say $R_k(i,j) = v \in \{0,1\}$.

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We make $g_{i,j,k} \leq_P r_v$ and $g_{i,j,l} \leq_P r_0, r_1$ for all $l \leq s$ not equal to k.

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Thus, in the end, all but one $g_{i,j,k}$ are below both r_0 and r_1, and exactly one is below just one of r_0 and r_1. The latter $g_{i,j,k}$ is below r_0 if $R(i,j)$ does not hold, and below r_1 otherwise.

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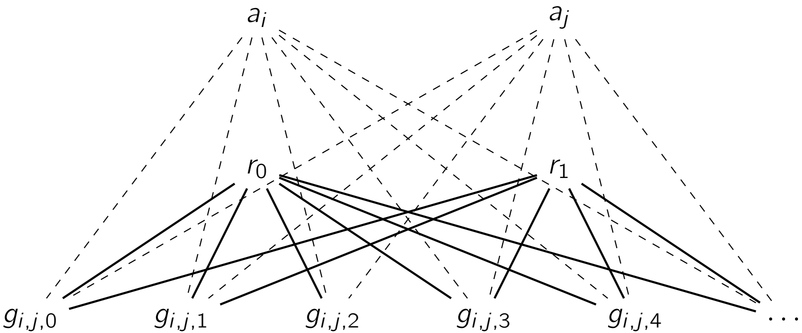

Any copy of R can compute a partial order of such order type, and from any such order we can compute a copy of R. □

Thank you for your attention.