An Introduction to Constructive Reverse Mathematics

12th Asian Logic Conference

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with Douglas Bridges and Maarten McKubre-Jordens

December 9, 2011

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- **BISH** is the most fundamental of these.

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- **WLPO**: For every binary sequence, either all the terms are equal to 0, or it is impossible for all the terms to be equal to 0.
- **LLPO**: For every binary sequence with **at most** one term equal to 1, either all the even terms are equal to 0, or all the odd terms are equal to 0.

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- For a path α, we denote by $\beta = \overline{\alpha}n$ the sequence consisting of the first n terms of α. We say that β is a **restriction** of α.
- A path α is **blocked** by $B \subseteq \mathbf{2}^*$ iff some restriction of α belongs to B.

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- $B \subseteq \mathbf{2}^*$ is a **bar** iff each infinite path of $\mathbf{2}^*$ is blocked by B.

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- $B \subseteq \mathbf{2}^*$ is a **uniform bar** iff, furthermore, there exists a number N such that each finite path of length N is blocked by B.

$$(\exists N \in \mathbf{N})(\forall u \in \mathbf{2}^*\colon |u| = N)(\exists n \leqslant N)\left[\overline{u} n \in B\right]$$

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- For example, the **fan theorem for detachable bars**:

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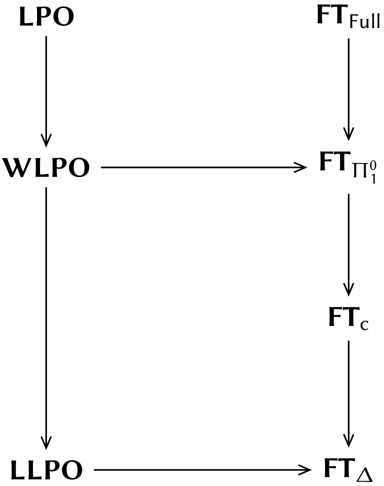
- A subset $B \subseteq \mathbf{2}^*$ is **detachable** iff, for each $u \in \mathbf{2}^*$, either $u \in B$ or $u \notin B$.

Omniscience Principles and Fan Theorems



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- We will make use of a predicate $\mathbf{Bl}^k_d(n)$ to mean that the left ($d = 0$) or right ($d = 1$) half of the k^{th} split of the binary fan is uniformly blocked by B at depth n.

$$\mathbf{Bl}^k_d(n) \equiv (\forall u\colon |u| = n \wedge \mathbf{StartsWith}(u, \overline{x}k^\frown d)) \\ (\exists m \leqslant n)\left[\overline{u}m \in B\right]$$

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- $\mathbf{AS}_{[0,1]}$ is equivalent to a version of the fan theorem, $\mathbf{FT}_c$.

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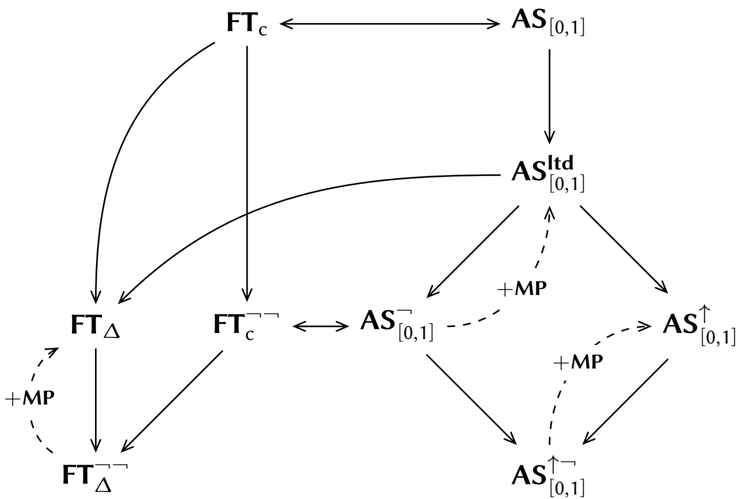
The Picture So Far



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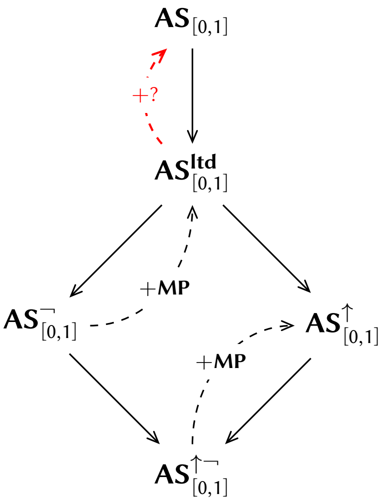

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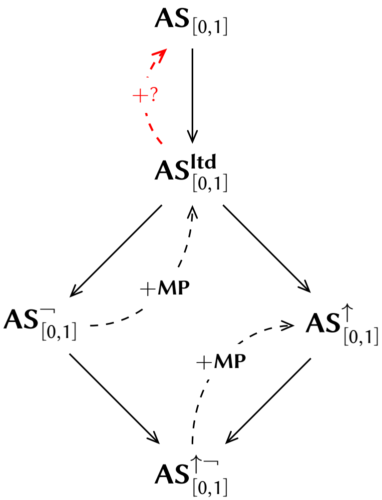

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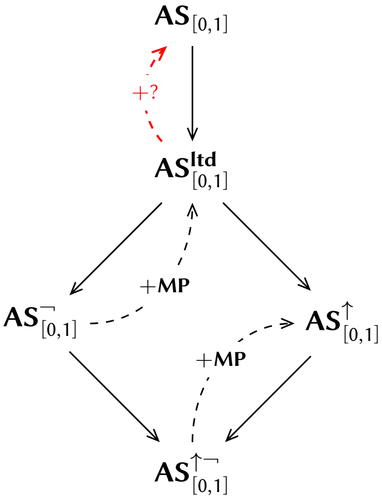

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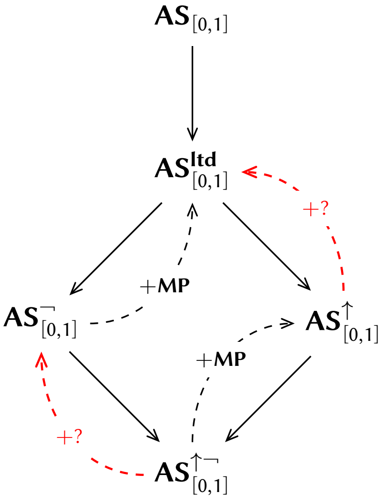

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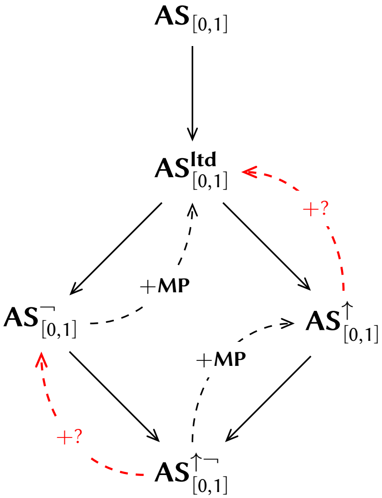

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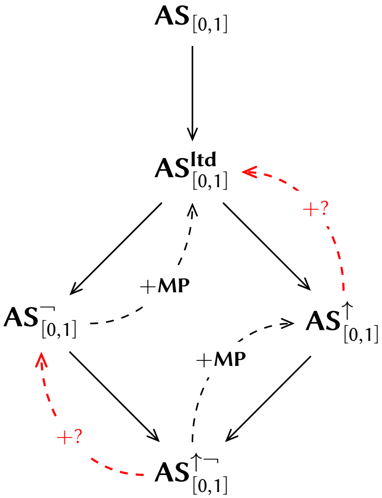

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- Need to somehow construct a nondecreasing Specker sequence from an arbitrary one.

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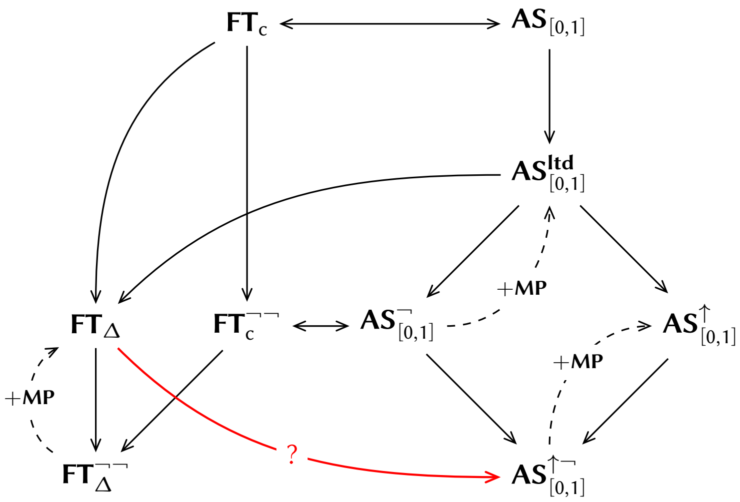

Acknowledgements

Thanks to:

- Douglas Bridges
- Maarten McKubre-Jordens
- The University of Canterbury

Selected references

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