An Introduction to Constructive Reverse Mathematics

12th Asian Logic Conference

James Dent
with Douglas Bridges and Maarten McKubre-Jordens

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What is constructive mathematics?

Existence means constructibility. Constructive proofs embody algorithms. We reject the law of excluded middle \( P \lor \neg P \).

Not a critique of classical mathematics but rather a programme of creating more contentful proofs.

Three main varieties: INT, RUSS, and BISH. BISH is the most fundamental of these.

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- Three main varieties: INT, RUSS and BISH.
- BISH is the most fundamental of these.
We sort noni and semiiconstructive principles into equivalence classes over $\mathbb{BISH}$. This is suggestive of augmented "semiiconstructive" systems.

We concern ourselves with three main families of principles:

1. Omniscience principles
2. Fan theorems
3. sntiiSpecker properties
Constructive reverse mathematics

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- Fan theorems
- Anti-Specker properties
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- **WLPO:** For every binary sequence, either all the terms are equal to 0, or it is impossible for all the terms to be equal to 0.
Omniscience Principles

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- **WLPO**: For every binary sequence, either all the terms are equal to 0, or it is impossible for all the terms to be equal to 0.
- **LLPO**: For every binary sequence with at most one term equal to 1, either all the even terms are equal to 0, or all the odd terms are equal to 0.
Fan Theorems: Terminology
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- A path in $2^*$ is just a finite or infinite binary sequence.  
- For a path $\alpha$, we denote by $\beta = \bar{\alpha}_n$ the sequence consisting of the first $n$ terms of $\alpha$. We say that $\beta$ is a restriction of $\alpha$.  

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Fan Theorems: Terminology

- Fan theorems concern subsets of the **complete binary fan**, $2^*$.  
- A **path** in $2^*$ is just a finite or infinite binary sequence.  
- For a path $\alpha$, we denote by $\beta = \bar{\alpha}_n$ the sequence consisting of the first $n$ terms of $\alpha$. We say that $\beta$ is a **restriction** of $\alpha$.  
- A path $\alpha$ is **blocked** by $B \subseteq 2^*$ iff some restriction of $\alpha$ belongs to $B$.  

Fan Theorems: Terminology
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- $B \subseteq 2^*$ is a **bar** iff each infinite path of $2^*$ is blocked by $B$.

$$\forall \alpha \in 2^{N^+} \exists n \in N \left[ \overline{\alpha}n \in B \right]$$
Fan Theorems: Terminology

- $B \subseteq 2^*$ is a **bar** iff each infinite path of $2^*$ is blocked by $B$.

  $$(\forall \alpha \in 2^{N^+})(\exists n \in N) \left[ \alpha n \in B \right]$$

- $B \subseteq 2^*$ is a **uniform bar** iff, furthermore, there exists a number $N$ such that each finite path of length $N$ is blocked by $B$.

  $$(\exists N \in N)(\forall u \in 2^*: |u| = N)(\exists n \leq N) \left[ \bar{u}n \in B \right]$$
Fan Theorems

For example, the fan theorem for detachable bars states

Every detachable bar of \( 2^* \) is uniform.

A subset \( B \subseteq 2^* \) is detachable if for each \( u \in 2^* \), either \( u \in B \) or \( u \notin B \).
Fan Theorems

- Brouwer’s **fan theorem for \( \mathcal{B} \)-bars** states:

  \textbf{FT}\( \mathcal{B} \): Every bar for \( 2^* \) with the property \( \mathcal{B} \) is a uniform bar.
Fan Theorems

- Brouwer’s fan theorem for \( \neg \)-bars states:
  \[
  \mathbf{FT}_\neg: \text{ Every bar for } 2^* \text{ with the property } \neg \text{ is a uniform bar.}
  \]

- For example, the fan theorem for detachable bars:
  \[
  \mathbf{FT}_\Delta: \text{ Every detachable bar of } 2^* \text{ is uniform.}
  \]
Fan Theorems

- Brouwer’s **fan theorem for ?-bars** states:

  \( \text{FT}_{\forall}: \) Every bar for \( 2^* \) with the property \( \forall \) is a uniform bar.

- For example, the **fan theorem for detachable bars**:

  \( \text{FT}_{\Delta}: \) Every detachable bar of \( 2^* \) is uniform.

- A subset \( B \subseteq 2^* \) is **detachable** iff, for each \( u \in 2^* \), either \( u \in B \) or \( u \notin B \).
Omniscience Principles and Fan Theorems

LPO \rightarrow FT_{\text{Full}}

WLPO \rightarrow FT_{\Pi_1^0}

LLPO \rightarrow FT_{\Delta}

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LLPO $\iff$ $\text{FT}_{\Delta}$
LLPO $\iff FT_{\Delta}$

- Let $B$ be a detachable bar.
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We show that $B$ is uniform by growing a maximal-length finite path $x$ that is not blocked by $B$. 
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We show that $B$ is uniform by growing a maximal-length finite path $x$ that is not blocked by $B$.

We will make use of a predicate $\text{Bl}^k_d(n)$ to mean that the left ($d = 0$) or right ($d = 1$) half of the $k^{th}$ split of the binary fan is uniformly blocked by $B$ at depth $n$.

$$\text{Bl}^k_d(n) \equiv (\forall u: |u| = n \land \text{StartsWith}(u, \bar{k}\bar{d}))$$

$$\quad (\exists m \leq n)[\bar{u}m \in B]$$

$$\neg\text{Bl}^k_d(n) \equiv (\exists u: |u| = n \land \text{StartsWith}(u, \bar{k}\bar{d}))$$

$$\quad (\forall m \leq n)[\bar{u}m \notin B]$$
LLPO $\iff \text{FT}_\Delta$
\textbf{LLPO} \implies \textbf{FT}_\Delta

- Suppose we have found the first $k$ terms of $x$. 

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**LLPO** \[\iff \ FT_\Delta\]

- Suppose we have found the first $k$ terms of $x$.
- Define a binary sequence $(a_n)$ such that:
  
  Invoking LLPO we see that either all the even terms of $(a_n)$ are zero in which case we set $x_{k+n} = n$ or all the odd terms of $(a_n)$ are zero in which case we set $x_{k+n} = mj$. 

\[ \text{LLPO} \iff \text{FT}_\Delta \]

- Suppose we have found the first \( k \) terms of \( x \).
- Define a binary sequence \( (a_n) \) such that:
  - \( a_n = 0 \) iff:
LLPO $\implies$ FT$\Delta$

- Suppose we have found the first $k$ terms of $x$.
- Define a binary sequence $(a_n)$ such that:
  - $a_n = 0$ iff:
    - a prior term of $(a_n)$ is 1, or
LLPO $\iff$ FT$_\Delta$

- Suppose we have found the first $k$ terms of $x$.
- Define a binary sequence $(a_n)$ such that:
  - $a_n = 0$ iff:
    - a prior term of $(a_n)$ is 1, or
    - $(n$ is even $\land \neg BL_1^k(n)) \lor (n$ is odd $\land \neg BL_0^k(n))$
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  - \( a_n = 1 \) iff:
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Invoking \textbf{LLPO}, we see that either:
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Specker’s theorem is a fundamental result in recursive mathematics. Its variation upon it states:

There exists a sequence \((z_n)\) in \([m, n]\) that is eventually bounded away from each point of \([m, n]\).

That is, for each \(x \in [m, n]\), there exist \(N\) and \(\delta > m\) such that

\[|z_n - x| > \delta\]

for all \(n \geq N\).

We call such a sequence a Specker sequence.

The well-known anti-Specker property can be formulated as follows:

\(A[m, n]\) if \((z_n)\) is a sequence in \([m, n] \cup \{\theta\}\) that is eventually bounded away from each point of \([m, n]\), then \(z_n = \theta\) eventually.

\(A[m, n]\) is equivalent to a version of the fan theorem.
Anti-Specker Properties

- **Specker’s theorem** is a fundamental result in recursive mathematics. A variation upon it states:

  **Speck**: There exists a sequence \((z_n)\) in \([0, 1]\) that is eventually bounded away from each point of \([0, 1]\).
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  \[\text{AS}_{[0,1]}: \text{If } (z_n) \text{ is a sequence in } [0, 1] \cup \{2\} \text{ that is eventually bounded away from each point of } [0, 1], \text{ then } z_n = 2 \text{ eventually.}\]
Anti-Specker Properties

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- \( \text{AS}_{[0,1]} \) is equivalent to a version of the fan theorem, \( \text{FT}_c \).
Anti-Specker Properties

The following weak anti-Specker properties are less well understood.

If $(z_n)$ is a sequence in $\mathbb{N} \cup \{0\}$ that is eventually bounded away from each point of $\mathbb{N}$, then there exists $k$ such that $z_k = 0$.

If $(z_n)$ is a nondecreasing sequence in $\mathbb{N} \cup \{0\}$ that is eventually bounded away from each point of $\mathbb{N}$, then $z_n = 0$ eventually.

If $(z_n)$ is a sequence in $\mathbb{N}$, then it is impossible for $(z_n)$ to be eventually bounded away from each point of $\mathbb{N}$.

If $(z_n)$ is a nondecreasing sequence in $\mathbb{N}$, then it is impossible for $(z_n)$ to be eventually bounded away from each point of $\mathbb{N}$.
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  - $\text{AS}_{[0,1]}^\uparrow$: If $(z_n)$ is a nondecreasing sequence in $[0, 1] \cup \{2\}$ that is eventually bounded away from each point of $[0, 1]$, then $z_n = 2$ eventually.
The following **weak anti-Specker properties** are less well understood:

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- **$AS_{[0,1]}^\uparrow$**: If $(z_n)$ is a **nondecreasing** sequence in $[0, 1] \cup \{2\}$ that is eventually bounded away from each point of $[0, 1]$, then $z_n = 2$ eventually.
- **$AS_{[0,1]}^\downarrow$**: If $(z_n)$ is a sequence in $[0, 1]$, then it is impossible for $(z_n)$ to be eventually bounded away from each point of $[0, 1]$. 

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Anti-Specker Properties

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  - **$\text{AS}_{[0,1]}^{\uparrow}$**: If $(z_n)$ is a **nondecreasing** sequence in $[0, 1] \cup \{2\}$ that is eventually bounded away from each point of $[0, 1]$, then $z_n = 2$ eventually.
  - **$\text{AS}_{[0,1]}^{\downarrow}$**: If $(z_n)$ is a sequence in $[0, 1]$, then it is impossible for $(z_n)$ to be eventually bounded away from each point of $[0, 1]$.
  - **$\text{AS}_{[0,1]}^{\uparrow \downarrow}$**: If $(z_n)$ is a **nondecreasing** sequence in $[0, 1]$, then it is impossible for $(z_n)$ to be eventually bounded away from each point of $[0, 1]$.
The Picture So Far

- FT_c
- AS_{0,1}
- AS_{0,1}^{\text{ld}}
- AS_{0,1}^{\uparrow}
- FT_{\Delta}
- FT_{c}^{\downarrow}
- AS_{0,1}^{\uparrow}
- +MP
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Where To Next?

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Where To Next?

$\text{AS}_{[0,1]} \xrightarrow{+?} \text{AS}_{\text{ltd}}^{[0,1]} \xrightarrow{ \text{+MP} } \text{AS}_{[0,1]}^{\uparrow} \xrightarrow{ \text{+MP} } \text{AS}_{[0,1]}^{\uparrow}$

What separates $\text{AS}_{[0,1]}$ and $\text{AS}_{\text{ltd}}^{[0,1]}$?

Need to somehow construct a nondecreasing Specker sequence from an arbitrary one.
Where To Next?

- $\text{AS}_{[0,1]}^{\text{ltd}} + \text{MP}$ is “close” to $\text{AS}_{[0,1]}$.

Diagram:

\[
\begin{array}{c}
\text{AS}_{[0,1]}^{\text{ltd}} \\
\uparrow \\
\text{AS}_{[0,1]}^{-} \\
\text{AS}_{[0,1]}^{\text{MP}} \\
\downarrow \\
\text{AS}_{[0,1]}^{\text{ltd}} \\
\downarrow \\
\text{AS}_{[0,1]}^{\uparrow} \\
\end{array}
\]
Where To Next?

- \( \text{AS}_{[0,1]} \) + MP is “close” to \( \text{AS}_{[0,1]} \).
- But MP doesn’t seem to be enough to close this gap.

\[ \begin{array}{c}
\text{AS}_{[0,1]} \\
\downarrow \\
\text{AS}_{[0,1]} \\
\downarrow
\end{array} \quad \begin{array}{c}
\text{AS}_{[0,1]} \\
\downarrow \\
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- But \( \text{MP} \) doesn’t seem to be enough to close this gap.

\[
\begin{align*}
\text{AS}_{[0,1]} & \quad \text{AS}_{[0,1]}^{\text{ltd}} \\
\text{AS}_{[0,1]}^{-} & \quad \text{AS}_{[0,1]}^{\uparrow} \\
\text{AS}_{[0,1]}^{\uparrow} & \quad \text{AS}_{[0,1]}^{\text{ltd}} \\
\end{align*}
\]

Need to somehow construct a nondecreasing Specker sequence from an arbitrary one.

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Where To Next?

- $\text{AS}_{[0,1]} + \text{MP}$ is “close” to $\text{AS}_{[0,1]}$.
- But MP doesn’t seem to be enough to close this gap.

- What separates $\text{AS}_{[0,1]}^\neg$ and $\text{AS}_{[0,1]}^\neg$, or $\text{AS}_{[0,1]}^\neg$ and $\text{AS}_{[0,1]}^\neg$?
Where To Next?

- $\text{AS}_{[0,1]}^{\text{ltd}} + \text{MP}$ is “close” to $\text{AS}_{[0,1]}$.
- But $\text{MP}$ doesn’t seem to be enough to close this gap.

- What separates $\text{AS}_{[0,1]}^{\leftarrow}$ and $\text{AS}_{[0,1]}^{\leftarrow \rightarrow}$, or $\text{AS}_{[0,1]}^{\text{ltd}}$ and $\text{AS}_{[0,1]}^{\rightarrow}$?
- Need to somehow construct a nondecreasing Specker sequence from an arbitrary one.

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Selected references

