Almost (MP)-based substructural logics

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$$\Gamma, \varphi \vdash_{\mathrm{CPC}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathrm{CPC}} \varphi \to \psi$$

$$\Gamma, \varphi \vdash_{\mathrm{S4}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathrm{S4}} \Box\varphi \to \psi$$

$$\Gamma, \varphi \vdash_{\mathrm{T}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathrm{T}} \Box \overset{n}{\cdots} \Box\varphi \to \psi \text{ for some natural } n$$

$$\Gamma, \varphi \vdash_{\mathrm{K}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathrm{K}} \bigwedge_{n \in A} \Box \overset{n}{\cdots} \Box \varphi \to \psi \text{ for some set } A \text{ of naturals}$$

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathrm{L}} \chi(\varphi, \psi) \text{ for some } \chi \in \Psi$$

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathrm{L}} \chi(\varphi) \to \psi \text{ for some } \chi \in \Psi$$

Substructural logics

Non-associative Full Lambek Calculus SL [Galatos-Ono, APAL, 2010]

$$\mathcal{L}_{\mathrm{SL}} = \{\backslash, /, \&, \wedge, \vee, \mathbf{1}, \mathbf{0}, \top, \bot\}$$

$$\vdash \varphi \backslash \varphi \qquad \varphi, \varphi \backslash \psi \vdash \psi \qquad \varphi \vdash (\varphi \backslash \psi) \backslash \psi$$

$$\varphi \backslash \psi \vdash (\psi \backslash \chi) \backslash (\varphi \backslash \chi) \qquad \psi \backslash \chi \vdash (\varphi \backslash \psi) \backslash (\varphi \backslash \chi)$$

$$\vdash \varphi \backslash ((\psi / \varphi) \backslash \psi) \qquad \varphi \backslash (\psi \backslash \chi) \vdash \psi \backslash (\chi / \varphi) \qquad \psi / \varphi \vdash \varphi \backslash \psi$$

$$\vdash \varphi \wedge \psi \backslash \varphi \qquad \vdash \varphi \wedge \psi \backslash \psi$$

$$\varphi, \psi \vdash \varphi \wedge \psi \qquad \vdash (\chi \backslash \varphi) \wedge (\chi \backslash \psi) \backslash (\chi \backslash \varphi \wedge \psi)$$

$$\vdash \varphi \backslash \varphi \vee \psi \qquad \vdash (\varphi \backslash \chi) \wedge (\psi \backslash \chi) \backslash (\varphi \vee \psi \backslash \chi)$$

$$\vdash \psi \backslash \varphi \vee \psi \qquad \vdash (\chi / \varphi) \wedge (\chi / \psi) \backslash (\chi / \varphi \vee \psi)$$

$$\vdash \psi \backslash (\varphi \backslash \varphi \,\&\, \psi) \qquad \psi \backslash (\varphi \backslash \chi) \vdash \varphi \,\&\, \psi \backslash \chi$$

$$\vdash \mathbf{1} \qquad \vdash \mathbf{1} \backslash (\varphi \backslash \varphi) \qquad \vdash \varphi \backslash (\mathbf{1} \backslash \varphi)$$

A convention

Convention

A logic L in a language $\mathcal{L}$ containing $\backslash$ or $/$ is substructural if

- L is an expansion of the $\mathcal{L} \cap \mathcal{L}_{\mathrm{SL}}$-fragment of SL.
- for each n, $i < n$, and each n-ary connective $c \in \mathcal{L} \setminus \mathcal{L}_{\mathrm{SL}}$:

$$\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{L}} c(\chi_1, \ldots \chi_i, \varphi, \ldots, \chi_n) \to c(\chi_1, \ldots \chi_i, \psi, \ldots, \chi_n),$$

where $\to$ is any of the implications in $\mathcal{L}$.

Let us fix an one of the implications and denote it as $\to$.

Examples of substructural logics

- substructural logics in Ono's sense including e.g. monoidal logic, uninorm logic, psBL, GBL, BL, Intuitionistic logic, (variants of) relevance logics, Łukasiewicz logic;
- non-associative substructural logics recently developed by Buszkowski, Farulewski, Galatos, Ono, Halaš, Botur, etc.
- expansions by additional connectives, e.g. (classical) modalities, exponentials in (variants of) Linear Logic and Baaz's Delta in fuzzy logics;
- fragments to languages containing implication, e.g. BCK, BCI, psBCK, BCC, hoop logics, etc.;

A problem?

- Is the logic $\mathrm{BCK}_\wedge$ of BCK-semilattices substructural?
- It does not satisfy $(\chi \setminus \varphi) \wedge (\chi \setminus \psi) \setminus (\chi \setminus \varphi \wedge \psi)$.
- Solution: it can be considered a substructural logic in our sense if formulated in the language $\{\setminus, \overline{\wedge}, \dots\}$.

Syntax: associativity and other notable extensions

Definition

FL is the extension of SL by

- $\vdash_{\mathrm{L}} \varphi \mathbin{\&} (\psi \mathbin{\&} \chi) \to (\varphi \mathbin{\&} \psi) \mathbin{\&} \chi$
- $\vdash_{\mathrm{L}} (\varphi \mathbin{\&} \psi) \mathbin{\&} \chi \to \varphi \mathbin{\&} (\psi \mathbin{\&} \chi)$

Axiomatic extensions of SL and FL

usual name	s	&-form	→-form
exchange	e	$\varphi \mathbin{\&} \psi \to \psi \mathbin{\&} \varphi$	$\varphi \to (\psi \to \chi) \vdash \psi \to (\varphi \to \chi)$
contraction	c	$\varphi \to \varphi \mathbin{\&} \varphi$	$\varphi \to (\varphi \to \psi) \vdash \varphi \to \psi$
weakening	w	i + o	
$\Downarrow$			
left-weak.	i	$\varphi \mathbin{\&} \psi \to \psi$	$\psi \to (\varphi \to \psi)$
right-weak.	o		$\mathbf{0} \to \varphi$

Hilbert style axiomatic systems

Theorem (Logics and only rules needed for their axiomatizations)

$\mathrm{FL_{ew}}$	*modus ponens*
K	*modus ponens and necessitation*
$\mathrm{FL_e}$	*modus ponens and unit adjunction*
FL	*modus ponens and product normality*

(MP)	$\varphi, \varphi \backslash \psi \vdash \psi$		modus ponens
(Adj)	$\varphi \vdash \varphi \wedge \mathbf{1}$		unit adjunction
(nec)	$\varphi \vdash \Box\varphi$		necessitation
(PN)	$\varphi \vdash \lambda_\alpha(\varphi)$	$\varphi \vdash \rho_\alpha(\varphi)$	product normality

Definition

- *a left conjugate* of φ is $\lambda_\alpha(\varphi) = (\alpha \backslash \varphi \,\&\, \alpha) \wedge \mathbf{1}$
- *a right conjugate* of φ is $\rho_\alpha(\varphi) = (\alpha \,\&\, \varphi \,/\, \alpha) \wedge \mathbf{1}$
- *an iterated conjugate* of φ is $\gamma_{\alpha_1}(\gamma_{\alpha_2} \cdots \gamma_{\alpha_n}(\varphi) \ldots))$
  where $\gamma_{\alpha_i} = \lambda_{\alpha_i}$ or $\gamma_{\alpha_i} = \rho_{\alpha_i}$

Almost (MP)-based logics

and

Almost-Implicational Deduction Theorem

Conventions

We fix

- a substructural logic L in language *with* $\rightarrow, \&,$ *and* $\mathbf{1}$
- a new propositional variable $\star$

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- formulae vs. $\star$-formulae
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Definition

Let δ be a $\star$-formula and φ a formula. We define $\star$-substitution σ as: $\sigma(\star) = \varphi$ and $\sigma(p) = p$ for $p \neq \star$. Then $\sigma\delta$ will be written as $\delta(\varphi)$ and it is a *formula.*

Main definition

Definition (Almost (MP)-based substructural logic)

L is *almost* (MP)*-based* w.r.t. a set bDT of $\star$*-formulae* (called *basic deduction terms*) if it has an axiomatic system where

- there are no rules with *three or more premises*
- there is only one rule with *two premises:* modus ponens
- the remaining rules are $\{\varphi \vdash \chi(\varphi) \mid \varphi \in \mathrm{Fm}, \chi \in \mathrm{bDT}\}$

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- the remaining rules are $\{\varphi \vdash \chi(\varphi) \mid \varphi \in \mathrm{Fm}, \chi \in \mathrm{bDT}\}$
- for each $\beta \in \mathrm{bDT}$ and each φ, ψ, there are $\beta_1, \beta_2 \in \mathrm{bDT}$ s.t.:

$$\vdash_{\mathrm{L}} \beta_1(\varphi \to \psi) \to (\beta_2(\varphi) \to \beta(\psi)).$$

Examples and conventions

Example

almost (MP)-based logics	possible bDTs
$\mathrm{FL_{ew}}$	$\emptyset$
$\mathrm{FL_e}$	$\{\star \wedge \mathbf{1}\}$
FL	$\{\lambda_\alpha(\star), \rho_\alpha(\star) \mid \alpha$ a formula$\}$
K	$\{\Box\star\}$

Examples and conventions

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K	$\{\Box\star\}$

Definition (Iterated and conjuncted Γ-formulae)

Let Γ be a set of $\star$-formulae. We define the sets of $\star$-formulae:

- Γ^* as the smallest set s.t.
  - $\star \in \Gamma^*$,
  - $\delta(\chi) \in \Gamma^*$ for each $\delta \in \Gamma$ and each $\chi \in \Gamma^*$.
- $\Pi(\Gamma)$ as the smallest set containing $\Gamma \cup \{\mathbf{1}\}$ and closed under $\&$.

Almost-Implicational Deduction Theorem

Theorem

Let L *be almost* (MP)*-based w.r.t. a set of basic deductive terms* bDT*. Then for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulae:*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathrm{L}} \delta(\varphi) \to \psi \textit{ for some } \delta \in \Pi(\mathrm{bDT}^*).$$

Examples

For each set $\Gamma \cup \{\varphi, \psi\}$ of formulae we have:

$\Gamma, \varphi \vdash_{\mathrm{FL}} \psi$ iff $\Gamma \vdash_{\mathrm{FL}} \mathop{\&}\limits_{\gamma \in A} \gamma(\varphi) \to \psi$ for some set A of iterated conjugates.

$\Gamma, \varphi \vdash_{\mathrm{FL_{ew}}} \psi$ iff $\Gamma \vdash_{\mathrm{FL_e}} \varphi \& \overset{n}{\cdots} \& \varphi \to \psi$ for some natural n.

$\Gamma, \varphi \vdash_{\mathrm{FL_e}} \psi$ iff $\Gamma \vdash_{\mathrm{FL_e}} \mathop{\&}\limits_{n \in A} ((\varphi \wedge \mathbf{1}) \overset{n}{\cdots} \wedge \mathbf{1}) \dots) \to \psi$ for some multiset A of naturals.

$\Gamma, \varphi \vdash_{\mathrm{K}} \psi$ iff $\Gamma \vdash_{\mathrm{K}} \mathop{\&}\limits_{n \in A} (\Box \overset{n}{\cdots} \Box \varphi) \to \psi$ for some multiset A of naturals.

Almost-Implicational Deduction Theorem cont.

Definition

A logic L has the *Almost-Implicational Deduction Theorem* w.r.t. a set of *deductive terms* DT, if for each set $\Gamma \cup \{\varphi, \psi\}$ of formulae:

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \qquad \text{iff} \qquad \Gamma \vdash_{\mathrm{L}} \delta(\varphi) \to \psi \text{ for some } \delta \in \mathrm{DT}.$$

Theorem

Let L *have the Almost-Implicational Deduction Theorem w.r.t.* DT.

- *If* L *is finitary, then it is almost* (MP)*-based w.r.t.*

  $$\mathrm{bDT} = \{\sigma\delta \mid \delta \in \mathrm{DT}, \sigma \text{ a } \star\text{-substitution such that } \sigma\star = \star\}.$$

- L *has the Almost-Implicational Deduction Theorem w.r.t.* $\mathrm{DT}' \subseteq \mathrm{DT}$ *IFF for every* $\delta \in \mathrm{DT}$ *and formula* φ *there is* $\delta' \in \mathrm{DT}'$ *s.t.* $\vdash_{\mathrm{L}} \delta'(\varphi) \to \delta(\varphi)$.

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Improved examples

Due to the contraction in K we obtain:

$$\Gamma, \varphi \vdash_{\mathrm{K}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathrm{K}} \bigwedge_{n \in A} (\Box \overset{n}{\cdots} \Box\varphi) \to \psi \text{ for some } \textbf{set } A \text{ of naturals.}$$

Also because T proves $\Box\varphi \to \varphi$ we obtain:

$$\Gamma, \varphi \vdash_{\mathrm{T}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathrm{T}} (\Box \overset{n}{\cdots} \Box\varphi) \to \psi \text{ for some } \textbf{natural } n$$

Also because $S4$ proves $\Box\varphi \to \Box\Box\varphi$ we obtain:

$$\Gamma, \varphi \vdash_{\mathrm{S4}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathrm{S4}} \Box\varphi \to \psi$$

Because $\mathrm{FL_e}$ proves $((\varphi \wedge \mathbf{1}) \overset{n}{\cdots} \wedge \mathbf{1}) \ldots) \to \varphi \wedge \mathbf{1}$ we obtain:

$$\Gamma, \varphi \vdash_{\mathrm{FL_e}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathrm{FL_e}} (\varphi \wedge \mathbf{1}) \& \overset{n}{\cdots} \& (\varphi \wedge \mathbf{1}) \to \psi \text{ for some natural } n.$$

Proof by Cases Property

Disjunction in Classical Logic

Proof by cases property

If $\Gamma, \varphi \vdash_{\mathrm{CPC}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{CPC}} \chi$, then $\Gamma, \varphi \vee \psi \vdash_{\mathrm{CPC}} \chi$.

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Dummett in '*The Logical Basis of Metaphysics*, HUP, 1991' says about (a weaker variant of) proof by cases:

> *If this law does not hold, the operator $\vee$ could not legitimately be called disjunction operator.*

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> *If this law does not hold, the operator $\vee$ could not legitimately be called disjunction operator.*

BUT this law does not hold in e.g. in K:

Indeed, from $\neg\varphi \vdash_{\mathrm{K}} \neg\varphi \vee \Box\varphi$ and $\varphi \vdash_{\mathrm{K}} \neg\varphi \vee \Box\varphi$ we get $\neg\varphi \vee \varphi \vdash_{\mathrm{K}} \neg\varphi \vee \Box\varphi$; i.e., $\vdash_{\mathrm{K}} \varphi \to \Box\varphi$

Proof by Cases Property

Theorem

Assume that L *has* $\vee$ *in its language, is almost* (MP)*-based logic w.r.t.* bDT*, and satisfies one of the following conditions:*

- L *is Rasiowa-implicative, i.e.* $\vdash_{\mathrm{L}} \varphi \to \mathbf{1}$
- *for each formula* φ *holds:*
  - $\vdash_{\mathrm{L}} \beta(\varphi) \to \mathbf{1}$ *for each* $\beta \in \mathrm{bDT}^* \setminus \{\star\}$ *and*
  - *there is* $\beta_0 \in \mathrm{bDT}$ *such that* $\vdash_{\mathrm{L}} \beta_0(\varphi) \to \varphi$*.*

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- *for each formula* φ *holds:*
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  - *there is* $\beta_0 \in \mathrm{bDT}$ *such that* $\vdash_{\mathrm{L}} \beta_0(\varphi) \to \varphi$*.*

Then the following meta-rule is valid in L*:*

$$\frac{\Gamma, \varphi \vdash_{\mathrm{L}} \chi \qquad\qquad \Gamma, \psi \vdash_{\mathrm{L}} \chi}{\Gamma \cup \{\alpha(\varphi) \vee \beta(\psi) \mid \alpha, \beta \in \mathrm{bDT}^*\} \vdash_{\mathrm{L}} \chi}\,.$$

Examples

Corollary

The following meta-rules are valid:

$$\frac{\Gamma, \varphi \vdash_{\mathrm{FL_{ew}}} \chi \qquad \Gamma, \psi \vdash_{\mathrm{FL_{ew}}} \chi}{\Gamma, \varphi \vee \psi \vdash_{\mathrm{FL_{ew}}} \chi}$$

$$\frac{\Gamma, \varphi \vdash_{\mathrm{FL_{e}}} \chi \qquad \Gamma, \psi \vdash_{\mathrm{FL_{e}}} \chi}{\Gamma, (\varphi \wedge \mathbf{1}) \vee (\psi \wedge \mathbf{1}) \vdash_{\mathrm{FL_{e}}} \chi}$$

$$\frac{\Gamma, \varphi \vdash_{\mathrm{FL}} \chi \qquad \Gamma, \psi \vdash_{\mathrm{FL}} \chi}{\Gamma \cup \{\gamma_1(\varphi) \vee \gamma_2(\psi) \mid \gamma_1, \gamma_2 \textit{ iterated conjugates}\} \vdash_{\mathrm{FL}} \chi}$$

$$\frac{\Gamma, \varphi \vdash_{\mathrm{K}} \chi \qquad \Gamma, \psi \vdash_{\mathrm{K}} \chi}{\Gamma \cup \{\Box^n(\varphi) \vee \Box^m(\psi) \mid n, m \geq 0\} \vdash_{\mathrm{K}} \chi}$$