A $\mathbf{GH}_1$ degree with the finite maximal chain property

Mingzhong Cai

Department of Mathematics
University of Wisconsin, Madison

December 16th 2011
12th Asian Logic Conference, Wellington

General Idea

We want to study the Turing degrees and give some "measure" on how high or how low the degree is.

Jump classes:

Definition

A degree $\mathbf{d}$ is low$_n$ $(\mathbf{L}_n)$ if $\mathbf{d}^{(n)} = \mathbf{0}^{(n)}$; $\mathbf{d}$ is high$_n$ $(\mathbf{H}_n)$ if $\mathbf{d}^{(n)} = \mathbf{0}^{(n+1)}$.

Definition

A degree $\mathbf{d}$ is generalized low$_n$ $(\mathbf{GL}_n)$ if $\mathbf{d}^{(n)} = (\mathbf{d} \vee \mathbf{0}')^{(n-1)}$, and $\mathbf{d}$ is generalized high$_n$ $(\mathbf{GH}_n)$ if $\mathbf{d}^{(n)} = (\mathbf{d} \vee \mathbf{0}')^{(n)}$.

They provide some rough measurement of whether the degree is close to $\mathbf{0}$ (or to $\mathbf{0}'$) by the number of iterations of jumps needed to collapse the degree to $\mathbf{0}$ (or to $\mathbf{0}'$).

Minimality:

Definition

A nonrecursive degree $\mathbf{d}$ is minimal if the interval $(\mathbf{0}, \mathbf{d})$ is empty. Relativizing this, a degree $\mathbf{d} > \mathbf{a}$ is a minimal cover of $\mathbf{a}$ if the interval $(\mathbf{a}, \mathbf{d})$ is empty.

Definition

We call a minimal cover of a minimal degree a 2-minimal degree. Similarly an $n + 1$-minimal degree is a minimal cover of an n-minimal degree. A degree has the finite maximal chain property if it is n-minimal for some $n \in \omega$.

That is, we measure the lowness of a degree $\mathbf{x}$ by the number of "minimality steps" from $\mathbf{0}$ to $\mathbf{x}$.

Remark

Study of minimality (both results and techniques) might be related to various definability results.

Theorem [Jockusch, Posner]

Every minimal degree is $\mathbf{GL}_2$ $(\mathbf{d}'' = (\mathbf{d} \vee \mathbf{0}')')$.

Proposition/Question [Lerman]

Every n-minimal degree below $\mathbf{0}'$ is $\mathbf{GL}_2$, hence $\mathbf{L}_2$. Is this true for n-minimal degrees in general?

Facts

Minimal degrees are "unpowerful":

- not r.e.
- not PA
- not ANR
- $\mathbf{GL}_2$
- not 1-generic
- not 1-random

One might ask whether n-minimal degrees are also unpowerful in some similar way.

Theorem [C.]

There is a 2-minimal degree which is ANR.

Question

Is there a 2-minimal or n-minimal degree which is PA?

Plan: build a 2-minimal degree which is $\overline{\mathbf{GL}_2}$.

Proposition

Suppose $\mathbf{a}$ is $\mathbf{GL_2}$ and $\mathbf{b}$ is a minimal cover of $\mathbf{a}$, then $\mathbf{b}$ is also $\mathbf{GL_2}$ if either of the following holds:

1. $\mathbf{b} < \mathbf{a}'$, or
2. $\mathbf{b}$ is hyperimmune-free relative to $\mathbf{a}$, i.e., every function f recursive in $\mathbf{b}$ is dominated by a function g recursive in $\mathbf{a}$ ($f(n) \leq g(n)$ for cofinitely many n).

This means that, in order to build an n-minimal degree which is not $\mathbf{GL}_2$, one has to use a relativized construction of "a hyperimmune minimal degree not recursive in $\mathbf{0}'$".

Our direct construction of a hyperimmune minimal degree

Basic tree component, block, with a guessing feature:

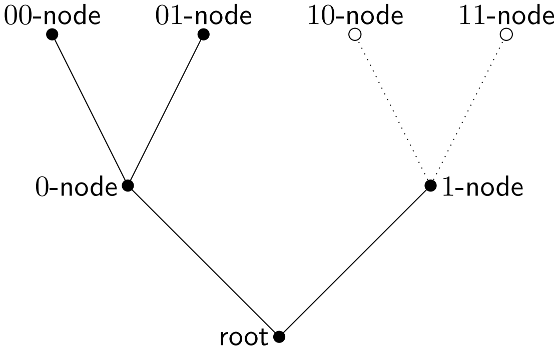

Figure: A guessing block

The 1-node has two successors if and only if some $\varphi_e(e)$ converges.

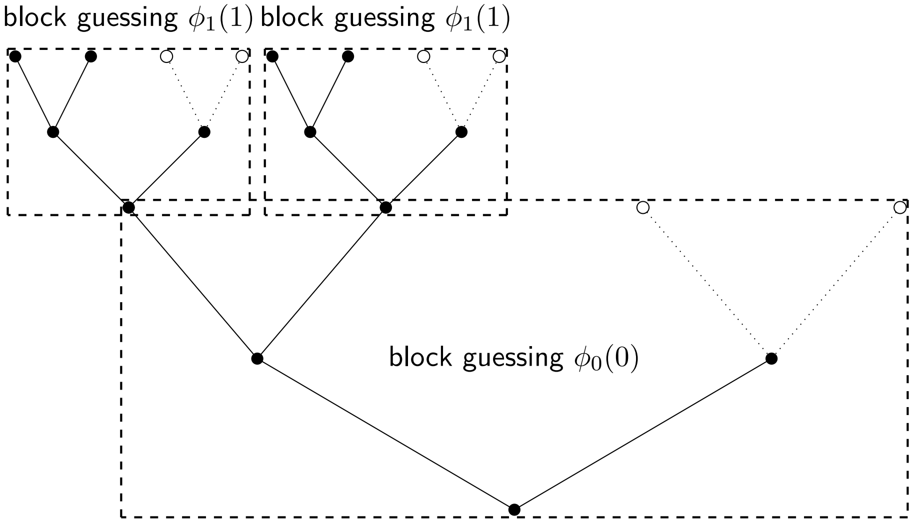

Figure: block structure on a tree

$\mathbf{ANR}$ 2-minimal

First step is to try to find $\mathbf{0} < \mathbf{a} < \mathbf{b}$ where $\mathbf{b}$ is ANR, i.e., there is a function recursive in $\mathbf{b}$ which is not dominated by m_K, the least modulus function of K ($m_K(n) = \mu s(K \upharpoonright n = K_s \upharpoonright n)$).

Idea:

Build A in $\Pi_{i \in \omega} i$ (a recursively bounded recursive tree) such that infinitely often $A(x)$ gives a correct guess at the number of changes in the recursive approximation to $m_K(x)$. In addition, $B(x)$ gives correct guess at whether $A(x)$ many changes can be found in the recursive approximation of $m_K(x)$.

However, building a $\overline{\mathbf{GL}_2}$ 2-minimal is more difficult.

Proposition

If $\mathbf{a}$ is $\mathbf{GL}_1$ and $\mathbf{b}$ is a minimal cover of $\mathbf{a}$, then $\mathbf{b}$ is $\mathbf{GL}_2$.

Proof.

$\mathbf{b}'' = (\mathbf{b} \vee \mathbf{a}')' = (\mathbf{b} \vee \mathbf{a} \vee \mathbf{0}')' = (\mathbf{b} \vee \mathbf{0}')'$. □

This means that we need to make $\mathbf{a}$ $\overline{\mathbf{GL}_1}$ if we want to make $\mathbf{b}$ $\overline{\mathbf{GL}_2}$.

Theorem [Sasso]

There is a minimal degree in $\overline{\mathbf{GL}_1}$.

Sasso used a notion of narrow subtrees to diagonalize against "$\mathbf{GL}_1$". "Path A is on narrow subtree of T" is $\Pi_1(A)$, and we can do some diagonalization:

$$\varphi_e^{A \oplus 0'} \neq A'.$$

$\overline{\mathbf{GL}_2}$ 2-minimal

Our plan: build A which is minimal and B which is a minimal cover of A such that for every e there is an x that

$$\varphi_e^{(B\oplus 0')'}(x) \neq B''(x).$$

$$\lim_s \varphi_e^{B\oplus 0'}(x,s) \neq (B \oplus A')'(x).$$

Left hand side:

force the limit to change until we cannot change it.

Right hand side:

find "narrow subtrees" which correspond to $(B \oplus A')'$ questions.

Key point:

When we switch to go off the "narrow subtree", how can we preserve the limit?

$\mathbf{GH}_2$

In order to make $B'' \equiv_T (B \oplus 0')''$, we want to decide $Tot(B \oplus 0')$, the totality problem for oracle $B \oplus 0'$, using B''.

Idea

In the construction, we try to force $\varphi_e^{B \oplus 0'}$ to be total step by step until we cannot continue (at which step we forced nontotality). Using a similar idea as in $\overline{\mathbf{GL}_2}$ construction, one can code this information of totality into some tree structure which we can retrieve using B''.

$\mathbf{GH}_1$

We want to make $B' \equiv_T (B \oplus 0')'$, i.e., use B' to decide the halting problem with oracle $B \oplus 0'$.

Idea

Use "partial trees" instead of "total trees" and make B' compute the whole construction (therefore decide whether we have forced the jump of $B \oplus 0'$.)

Note that this is the highest jump class we can reach by finite iterations of minimality.

Possible Future Work

- relations between n-minimality and other properties such as PA or random
- cones of "relative 2-minimality (with some extra restraint)"

Thank you!