Computability of Strongly Minimal Groups

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Recursive structures

Definition

- All languages L are countable and recursive.
- An L-structure A is recursive if $|A| = \omega$ and the atomic diagram of A is recursive.
- An L-structure A is decidable if $|A| = \omega$ and the elementary diagram of A is recursive.
- A is recursively (decidably) presentable if A is isomorphic to a recursive (decidable) model.

A useful observation about recursive structures

Observation

If $B \subseteq A$, A is recursive, and the universe of B is a Σ_1 subset of ω, then B is recursively presentable.

Proof.

Build the structure with universe ω where n represents the element of A which is the n^{th} to enter B. □

Theories with Recursive Models

- If T is recursive, then it has at least one decidable model (Henkin's construction).
- If A is recursive, then $\mathrm{Th}(A) \leq_T \mathrm{Th}(\mathbb{N}, +, \cdot)$, but need not be simpler.
- Even for strongly minimal or $\aleph_0$-categorical theories with all recursively presentable models, there can be no nicer bound.

Strongly Minimal Theories

Definition

A first order theory T is strongly minimal if for every $\bar{a} \in M \models T$ and every formula $\phi(x, \bar{y})$, $\phi(x, \bar{a})$ defines a finite or co-finite subset of M.

Example

- A regular acyclic graph with finite valence (say, the Cayley graph of $\mathbb{F}_2$);
- A vector space (say, $(\mathbb{Q}, +)$);
- An algebraically closed field, (say $(\mathbb{C}, +, \cdot)$)

Recursive Spectra of a Theory

In each of these examples, there is a notion of closure and dimension which characterizes models. This is not a coincidence.

Theorem (Baldwin-Lachlan)

If T is $\aleph_1$-categorical, then each model of T is determined by a single cardinal invariant, its dimension. If M is countable, then $\dim(M) \in \omega + 1$.

The following introduces the object we will be asking about:

Definition

If T is $\aleph_1$-categorical, define $SRM(T) = \{k \in \omega + 1 |$ the $k-$dimensional model is recursively presentable$\}$

The Spectrum Problem (unfortunately named)

Question

Which subsets of $\omega + 1$ are spectra? i.e., for which $S \subseteq \omega + 1$ does there exist some T so that $SRM(T) = S$?

Question

Which subsets of $\omega + 1$ are spectra of theories in finite languages?

Known Examples of Spectra

Answer

The following sets are known to be spectra:

- $\emptyset$
- $\omega + 1$
- $\{0\}$ (Goncharov 1978)
- $\{0, \ldots n\}$ (Kudaibergenov 1980)
- ω (Khoussainov, Nies, Shore 1997)
- $\omega + 1 \setminus \{0\}$ (Khoussainov, Nies, Shore 1997)
- $\{1\}$ (Nies 1999)
- $[1, \alpha)$ (Nies, Hirschfeldt unpublished)
- $\{\omega\}$ (Hirschfeldt, Khoussainov, Semukhin, 2006)
- $\{0, \omega\}$ (A.)

Known Examples of Spectra in Finite Languages

Answer

The following sets are known to be spectra in finite languages:

- $\emptyset$
- $\omega + 1$
- $\{0\}$ (Herwig, Lempp, Ziegler 1991)
- $\{0, \ldots n\}$ (A.)
- ω (A.)

For these results, I needed a Hrushovski construction, while each result on the last slide (aside from $\{0, \omega\}$) and $\{0\}$ here was in a disintegrated theory.

The Zilber Trichotomy

Zilber conjectured that every strongly minimal theory was of one of three types:

- Disintegrated (Essentially binary)
- Locally Modular (Essentially a vector space)
- Field-like (Essentially an algebraically closed field)

Theorem (Hrushovski 1991)

The Zilber trichotomy is false. There are non-trichotomous theories, and there are Hrushovski constructions!

An (Outlandish?) Conjecture

Conjecture

If T is a strongly minimal trichotomous theory in a finite language, then $SRM(T) = \emptyset, \omega + 1,$ or $\{0\}$.

Some evidence for the conjecture comes from the following theorems.

Theorem (A.-Medvedev)

If T is a disintegrated strongly minimal theory in a finite language, then $SRM(T) = \emptyset, \omega + 1,$ or $\{0\}$.

Theorem (A.-Medvedev)

If T is a locally modular theory in a finite language which expands a group, then $SRM(T) = \emptyset, \omega + 1,$ or $\{0\}$.

Reductions in the Disintegrated Case

Lemma

If T is disintegrated and strongly minimal in a finite language L, then there is a strongly minimal theory T' in a finite language L' comprised of only Rank 1 and Rank 0 relation symbols so that T and T' are interdefinable (i.e., there exists a definitional expansion of both).

But we need more for the reduction to respect the recursiveness of the models!

Lemma

If T is disintegrated and strongly minimal in a finite language L, then there is a strongly minimal theory T' in a finite language L' comprised of only Rank 1 and Rank 0 relation symbols so that T and T' are Δ_1-interdefinable (i.e., there exists a Δ_1-definitional expansion of both).

Lemma

If T is disintegrated strongly minimal in a finite language comprised of all Rank 1 and Rank 0 relation symbols, then $SRM(T) = \emptyset, \omega + 1,$ or $\{0\}$.

Recursive Reductions in the Disintegrated Case

Lemma

If T is disintegrated and strongly minimal in a finite language L, then there is a strongly minimal theory T' in a finite language L' comprised of only Rank 1 and Rank 0 relation symbols so that T and T' are Δ_1-interdefinable (i.e., there exists a Δ_1-definitional expansion of both).

Lemma

If T is disintegrated strongly minimal in a finite language comprised of all Rank 1 and Rank 0 relation symbols, then $SRM(T) = \emptyset, \omega + 1,$ or $\{0\}$.

What do strongly minimal expansions of groups look like?

Theorem (Reineke, 1975)

If $(G, \cdot)$ is a strongly minimal group, then G is abelian and either $G = \mathbb{Q}^\alpha \oplus \oplus_p \mathbb{Z}_{p^\infty}^{\alpha_p}$ with α_p finite OR $G = \mathbb{Z}_p^\beta$ for infinite β.

In the case that $(G, +, \ldots)$ is locally modular, the extra structure of the non-+ relations is not too bad. In fact, it is a quasi-vector-space over a division ring.

Quasiendomorphisms

- Let $(G, +, \ldots)$ be a strongly minimal modular group.
- Then $G_0 = \mathrm{acl}_G(\emptyset)$ is a subgroup of G.
- Let Q be the collection of connected $\mathrm{acl}(\emptyset)$-definable Rank 1 subgroups $K < G \times G$ with projection $\pi_1 : K \to G$ onto.
- Define C to be the collection of quasiendomorphisms of the form $G \times F$ for a finite $F \leq G$.
- Then $D = Q/C$ forms a division ring and G/G_0 is an D-vector-space.
- In fact, $G \cong G/G_0 \oplus G_0$ as a Q-quasi-vector-space.

Recursive Quasiendomorphisms

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- Then $D = Q/C$ forms a division ring and G/G_0 is an D-vector-space.
- In fact, $G \cong G/G_0 \oplus G_0$ as a Q-quasi-vector-space.
- If only we could get our hands on G_0 and D recursively!

Lemma

Given $(G,+,R_1,\dots R_n)$, there is a Δ_1-interdefinable theory T' in language $(G,+,H_1,\dots H_m)$ where each H_i is a group.

Lemma

Given $(G,+,H_1\dots H_m)$ where each H_i is a group, we can Δ_1-interdefinably replace each H_i by its connected subgroup.

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Given $(G,+,H_1,\dots H_m)$, where each H_i is a connected group, we can Δ_1-interdefinably replace each H_i by a sequence of rank 0 and rank 1 groups.

Recursively Taming Quasiendomorphisms

Lemma

Given $(G, +, R_1, \dots R_n)$, there is a Δ_1-interdefinable theory T' in language $(G, +, H_1, \dots H_m)$ where each H_i is a group.

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Recursively Taming Quasiendomorphisms

Lemma

Given $(G, +, R_1, \ldots, R_n)$ a modular group, there is $(G', +, H_1, \ldots, H_m)$ where each H_i is a quasiendomorphism so that G' is Δ_1-definable from G, and they have the same quasi-endomorphism ring D and the same $acl(\emptyset)$.

Lemma

If $(G, +, H_1, \ldots H_n)$ is a positive-dimensional modular group where each H_i defines a quasiendomorphism, then the quasiendomorphism ring D is recursively generated by the H_i's. Since $acl(\emptyset) = \cup_{d \in C} \operatorname{im}(d)$, it is a Σ_1 subset of the universe, thus a recursive structure.

Recursively Taming Quasiendomorphisms

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Putting it together

Theorem

If $(G, +, R_1, \ldots R_n)$ is a recursive modular strongly minimal group of positive dimension, then G_0 and D are recursive.

Corollary

If T is a stongly minimal theory of a modular group, then $SRM(T) = \emptyset, \omega + 1, \text{ or } \{0\}$.

Proof.

If there is a model of positive dimension, then both G_0 and D are recursively presented. From a recursive presentation of D, we can recursively present D^k, the D-vector space of dimension k for any k.

Let G be a model of dimension k ($k \in \omega + 1$), then $G \cong G_0 \oplus G/G_0 \cong G_0 \oplus D^k$. This gives a recursive presentation of G. □

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Thank you

ever so much for your patience!