School of Mathematics and Statistics
Te Kura Mātai Tatauranga

Math 466/467: Notes on special relativity

Honours-level module in Applied Mathematics

Matt Visser
School of Mathematics and Statistics,
Victoria University of Wellington,
Wellington, New Zealand.
E-mail: matt.visser@sms.vuw.ac.nz
URL: http://www.victoria.ac.nz/sms/about/staff/matt-visser

Draft form: 29 February 2016; LaTeX-ed February 29, 2016; © Matt Visser

Warning:

Semi–Private — Not intended for large-scale dissemination.
These notes are provided as a reading module for Math 466/467:
Honours-level Applied Mathematics.
There are still a lot of rough edges.
Semi–Private — Not intended for large-scale dissemination.

Contents

Part I

Notes

Chapter 1

The special relativity

Most of the special relativity, especially the foundations, can adequately be treated using matrix algebra — this allows us to put off the use of so-called "index gymnastics" for a little while. I'll introduce "index gymnastics" when it becomes useful to do so.

Indeed, most of the special relativity can easily be handled even more simply using high school algebra. This is both a blessing and a curse. Specifically, it means that any random nutter capable of getting high school algebra wrong can and will develop his/her own theory in counterpoint to special relativity — and will then insist on telling you all about it *at length*.

Comment: The two appendices have been added to show you that there is still serious research that can be done regarding special relativity — you will not be expected to answer any questions based on the appendices. $\diamondsuit$

1.1 Minkowski spacetime

Mathematically, the special relativity is synonymous with the geometry of Minkowski spacetime:

Definition 1
Minkowski spacetime is the pair $(\mathbb{R}^4, \eta)$, that is, $\mathbb{R}^4$ equipped with a distinguished quadratic form

$$\eta \equiv \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} = \eta^{-1}$$

Essentially everything in special relativity can be extracted from this definition, but without a little physics background you will have no idea why this definition was chosen.

First, why $\mathbb{R}^4$? Simply because to specify the location of an event you need to specify both the place (requiring three real parameters x, y, z), and the time (requiring one real parameter t). It is convenient (but not logically necessary) to use Cartesian coordinates in this four-dimensional space+time.

You could, for instance, use spherical polar coordinates on space

$$(x, y, z) \leftrightarrow (r, \theta, \phi)$$

but doing so does not gain you anything and makes life much messier. For instance, in spherical polar coordinates (t, r, θ, ϕ) the distinguished quadratic form becomes

$$\eta \equiv \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +r^2 & 0 \\ 0 & 0 & 0 & +r^2 \sin^2\theta \end{bmatrix}$$

with

$$\eta^{-1} \equiv \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +r^{-2} & 0 \\ 0 & 0 & 0 & +r^{-2} \sin^{-2}\theta \end{bmatrix} \neq \eta.$$

It is sobering to realise that at least some of the nut cases out there already lose contact with reality at this stage of the game — simply by not being able to handle the coordinate conversion from cartesian $\iff$ spherical polar coordinates.

Next we use the *central experimental fact* of special relativity:

Experimental fact: [I]
The speed of light (denoted c) is the same for all observers. $\diamondsuit$

Because I am teaching this course in a Mathematics department I will not go into a long song and dance routine as to how good the experimental evidence is for this statement. Suffice it to say that essentially all observational data is in its favour. This does not mean the physics community has given up looking for violations of special relativity, but we know that if such violations exist then they are definitely either extremely small or hiding in small ill-explored regions of parameter space. A good recent summary of the experimental situation (as of 2005) is:

- David Mattingly. Modern tests of Lorentz invariance.
  Living Reviews in Relativity **8** (2005) 5.
  http://relativity.livingreviews.org/Articles/lrr-2005-5/

For the purposes of this course I will take special relativity as being an exact description of empirical reality. (Looking at possible alternatives can wait until you understand the "standard" point of view.)

Project:
For an example of what happens if you abandon the relativity principle, see for instance appendix B. (The good news is that I do not expect you to do more than glance at this appendix, I will not assign any assignment problems based on this appendix.) If you are extremely bold you could try to extend this appendix in various ways. ♢

This (first) central experimental fact lets us do two things:

- Define $x^0 \equiv ct$, so that $X^\mu \equiv (x^0; \vec{x}) = (ct, x, y, z)$, and all the components of X^μ have the same dimensions. This is a convenience, and is not truly fundamental. The key point here is merely that you have *some* observer-independent conversion constant available that lets you convert time differences into distances...[1,2]
- Justify the definition and introduction of the quadratic form η.

To see how this justification for the introduction of η works, suppose we have two events X_1 and X_2 such that a light ray passes from event 1 to event 2. Define the difference

$$\Delta X \equiv X_2 - X_1 \equiv (c\Delta t; \Delta\vec{x})^T$$

so that ΔX is for convenience viewed as a column vector (in fact, a column 4-vector). Now by definition, since a light ray passes from event 1 to event 2

$$\frac{|\Delta\vec{x}|}{|\Delta t|} = c$$

But this means

$$|\Delta\vec{x}| = c\,|\Delta t|$$

which we can write in any of the equivalent forms

$$|\Delta\vec{x}| = |\Delta x^0|$$

$$|\Delta\vec{x}|^2 = (\Delta x^0)^2$$

$$-(\Delta x^0)^2 + |\Delta\vec{x}|^2 = 0$$

[1] I could of course use a different conversion constant with the dimensions of velocity, such as the speed of sound in air or water at STP [standard temperature and pressure]. Mathematically there is nothing exactly "wrong" with this, it's just physically awkward, and unhelpful. It's simpler and nicer all round to use the speed of light as your conversion constant.

[2] There are also some rather foul things around called VSL theories [variable speed of light]. Most of the VSL theories are internally inconsistent on purely mathematical grounds. Don't go there...

Finally we define the quadratic form

$$\eta(\Delta X, \Delta X) \equiv (\Delta X)^T \;\; \eta \;\; \Delta X = 0.$$

That is, we have proved:

Lemma 1
If we define $\Delta X \equiv X_2 - X_1$ then the mathematical statement that $\eta(\Delta X, \Delta X) = 0$ is equivalent to the physical statement that a light ray can pass from $1 \to 2$ (or $2 \to 1$).

This is why the quadratic form η is useful — it characterizes the possible paths of light rays in Minkowski spacetime.

If we pick a specific fixed point X_1 and look at the set

$$C(X_1) = \left\{ X_2 \;\middle|\; \eta(\Delta X, \Delta X) = 0 \right\},$$

then $C(X_1)$ is a double cone with apex at X_1 called the light-cone.

Exercise: [Trivial] Verify that

$$\eta^{-1} = \eta,$$

in the sense of η^{-1} being the inverse matrix for η. ♢

Exercise: Verify that $C(X_1)$ as defined above really is (both geometrically and topologically) a double cone with apex at X_1. ♢

1.2 The Lorentz and Poincare groups

Now what happens if two different observers compare notes? Galileo and Newton already taught us that two different observers do not have to agree on the specific labels $X = (ct; \vec{x})$ they give to a particular event, but there should be some rule for taking one observer's measurements and converting them to the other observer's conventions...

Ultimately this is nothing more mysterious than doing a coordinate transformation...

Because we have adopted Cartesian coordinates there is one nice simplification, linearity. If I put two identical objects next to each other, then their combined width is twice their individual widths. If I now ask a second distinct observer what he/she sees, one should still see two identical objects next to each other with a combined width that is twice their individual widths. Similarly for two ticks on a clock; two ticks is twice as long as one

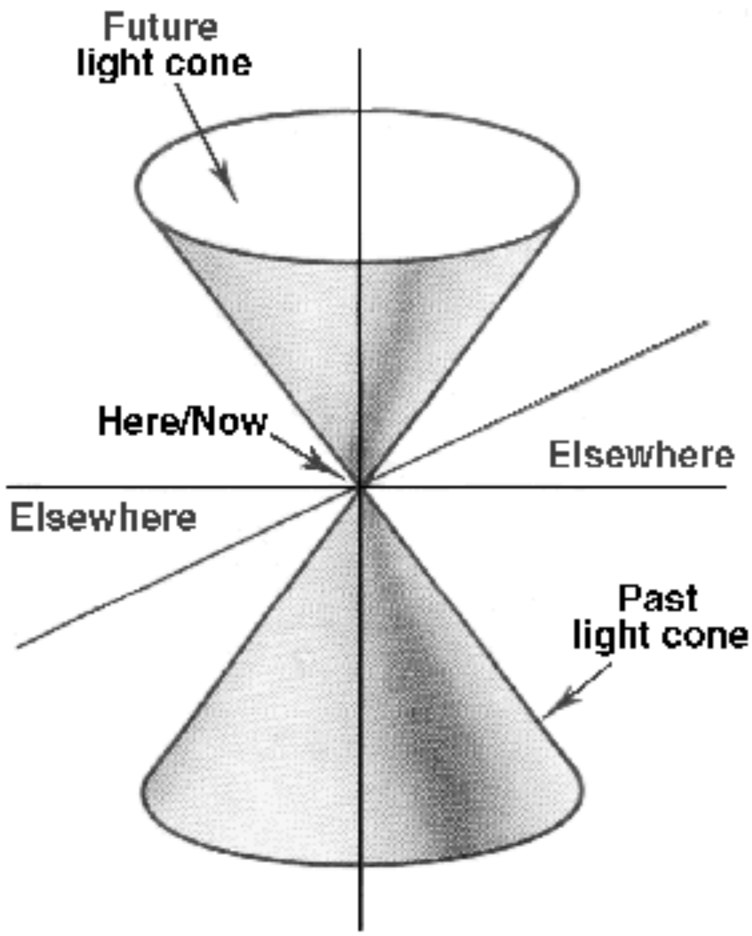

Figure 1.1: Light cone in special relativity. Time runs vertically up the page.

tick, and even if another observer disagrees on how long each individual tick takes, he/she should at least agree that two ticks takes twice as long as one tick.

That is: when comparing two different observers the transformation should be linear:[3]

$$X' = L\, X + a$$

Here the symbol a corresponds to a shift in the zero of coordinates (a translation) and L is some 4×4 matrix. But L cannot be an arbitrary 4×4 matrix.

1.2.1 Full Lorentz group:

Suppose events 1 and 2 are connected by a light ray, so that $\eta(\Delta X, \Delta X) = 0$. Now from the point of view of the second observer the two events are separated by

$$\Delta X' = L\, \Delta X$$

Now use the (first) central observational fact of special relativity: the speed of light is the same for all observers. This means that observer 2 has to agree that in terms of his measurements $\eta(\Delta X', \Delta X') = 0$. But this means $\eta(L\ \Delta X, L\ \Delta X) = 0$. So we have shown:

Lemma 2
If L is the matrix representing the coordinate change that arises in special relativity when two different observers compare notes, then

$$\forall \Delta X \qquad (\Delta X)^T\, \eta\, \Delta X = 0 \qquad \Rightarrow \qquad (\Delta X)^T\, [L^T\, \eta\, L]\, \Delta X = 0.$$

Exercise: One obvious solution to the above is

$$L^T\, \eta\, L \propto \eta$$

Prove that this is the *only* solution. ♢

If we take this exercise as proven (and this is an easy mathematical exercise, you can prove it with simple algebra and no additional physics input) then for some $\infty \neq \Omega \neq 0$:

$$L^T\, \eta\, L = \Omega^2\, \eta.$$

[3]This linearity would of course fail if I were to be so foolish as to use spherical polar coordinates. In that situation I would first have to convert the spherical polar coordinates to cartesian coordinates, apply the linear transformation in cartesian coordinates, and then convert back to spherical polar coordinates. There are good mathematical reasons why no-one in their right mind ever shows you the Lorentz transformations in spherical polar coordinates.

In fact $\Omega = 1$, but this needs a little more input than just the light cone structure (which is the only thing we have used so far).

Experimental fact: [II]
All observers are equivalent, there is no distinguished observer who is more special than any of the others.
(This is often called the "principle of relativity".) ◇

Warning: This means that all observers are "equivalent" in the sense that they are locked up in a lab and are not allowed to look outside; no fair looking at the fixed stars or the cosmic microwave background.[4] We are using the fact that *localized* observers (closed-box observers) cannot distinguish constant velocity motion from "at rest"; indeed in special relativity there is no universal meaning to the phrase "at rest" one always has to say "at rest *relative* to some particular observer" — this is where the phrase "relativity theory" comes from. ◇

- **Galileo**'s principle of relativity was the observation that you could not find any distinguished observer using purely the techniques of mechanics (kinematics, collisions, etc...). In a smoothly operating elevator you often cannot tell if you are moving until the acceleration/ deceleration stage. On a smooth ocean you cannot tell you are moving until the ship either deliberately changes velocity or hits some waves.

- **Einstein**'s principle of relativity was the empirical observation that this inability to detect absolute motion applied also to electromagnetic phenomena (light, radio, gamma rays). Indeed it is an *experimental fact* that this relativity principle applies to all of known physics.

Warning: The usefulness of the "Einstein relativity principle" ultimately depends on going out and making *experimental measurements*. The "principle" is simply a convenient way of summarizing a large number of *experimental facts*. ◇

Warning: The "Einstein relativity principle" does *not* imply that there's no such thing as "absolute rest"; it does however imply that the hypothetical state of "absolute rest" is not experimentally detectible by any means. In other language, the "Einstein relativity principle" does not imply that the "aether" does not exist, it implies that the "aether" is undetectable. It is then a physics judgment, based on Ockham's razor, to dispense with the aether as being physically irrelevant.

[4] This is another place where people often get confused, and where some of the nutters lose contact with empirical reality.

(Failure to recognize this elementary point is the source of at least one big branch of the crackpot literature.) $\diamondsuit$

Warning: There is currently [2016] some serious and significant work going on regarding the possible breakdown of Lorentz invariance at ultra-high energies (near the Planck scale where quantum gravity is expected to come into play). Please do not start playing with these rather speculative ideas until you have a good grasp of the standard theory. $\diamondsuit$

Now, because of this observed equivalence of observers, the quantity Ω appearing in the equation

$$L^T \eta L = \Omega^2 \eta$$

can *at most* be a function of the relative velocities of the two observers, and in particular it is immediately clear that for zero relative velocity $\Omega(0) = 1$. More generally, if $\Omega(v)$ is not a constant, its maxima and minima could be used to define privileged observers, as could any other special condition such as $\Omega = 1$ or, for instance, $\Omega = \sqrt{2}$. The only way to prevent privileged observers is if $\Omega \equiv 1$ for *all* relative velocities.

- For more details on this sort of argument, see the book "*Axiomatic Relativity*" by Hans Reichenbach. In this course, it would be overkill for us to present a full axiomatic derivation of special relativity — but I will point you to where to find such a discussion if you are interested.
- Remember — in the "axiomatic approach" to special relativity the axioms are simply a convenient distillation of a vast number of experimental facts.[5]

Exercise: Convince yourselves that in general

$$L(-v) = [L(v)]^{-1}$$

What assumptions do you need to justify this? $\diamondsuit$

Exercise: Using $L(-v) = [L(v)]^{-1}$ convince yourselves that

$$\Omega(v)\, \Omega(-v) = 1$$

$\diamondsuit$

Exercise: Using the second experimental fact, convince yourselves that

$$\Omega(v) = \Omega(-v)$$

[5]And failure to recognize this elementary point underlies yet another branch of crackpot rantings.

$\diamondsuit$

Exercise: Combining the above, convince yourselves that

$$\Omega(v) \equiv 1$$

$\diamondsuit$

The result $\Omega \equiv 1$ now reduces the constraint condition on the transformation matrices to

$$L^T \; \eta \; L = \eta.$$

In fact:

Theorem 1
The set $\{L\}$ of matrices corresponding to the coordinate change between two observers, constrained only by the relativity principle and the constancy of the speed of light, is completely characterized by

$$L^T \; \eta \; L = \eta.$$

This condition defines a group, in fact

$$\{L\} = O(3,1).$$

Here $O(3,1)$ denotes the group of pseudo-orthogonal matrices of signature $-+++$. This is the definition of the (full) Lorentz group; the matrices L are called Lorentz transformations.

Exercise: Explicitly verify that the group axioms are satisfied. $\diamondsuit$

Exercise: Explicitly verify that for any Lorentz transformation as defined above

$$\det[L] = \pm 1.$$

$\diamondsuit$

Exercise: Explicitly verify that for any Lorentz transformation as defined above

$$L^{-1} = \eta \; L^T \; \eta.$$

$\diamondsuit$

Historical note: If Einstein discovered special relativity, why are these transformations called Lorentz transformations? Because Lorentz did in fact first discover them, and wrote them down explicitly a couple of years before Einstein, but he did not realise what they meant — for Lorentz they were a peculiar mathematical symmetry he discovered buried deep in the Maxwell equations of electromagnetism. He did not understand many of the important physical implications.

Still, if Einstein had not come up with special relativity in 1905, there is widespread agreement in the physics community that Lorentz, Poincare, and/or possibly FitzGerald would have developed something very similar within a few years.

General relativity in contrast is much trickier. There is widespread agreement in the physics community that if Einstein had not incrementally developed general relativity over the period 1910-1915, then it would probably have taken many decades for someone else to develop a relativistic theory of gravitation. ♢

It is often useful to restrict the Lorentz group. Note that, as defined so far, it includes both parity transformations (mirror reflections along any axis) *and* time reversal:

$$x \to -x; \qquad L = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}$$

$$y \to -y; \qquad L = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}$$

$$z \to -z; \qquad L = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$t \to -t; \qquad L = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}$$

Exercise: Explicitly verify that for any 4-vector $N = (0, \mathbf{n})$, where $\mathbf{n}$ is any 3-space unit vector satisfying $||\mathbf{n}|| = 1$, the matrix

$$L = I_4 - 2N \otimes N$$

is a Lorentz transformation in the general sense described above.
How would you physically interpret these transformations? ♢

Exercise: Explicitly verify that the three matrices

$$\begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}; \qquad \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}; \qquad \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix};$$

are all Lorentz transformations in the general sense described above.
How would you physically interpret these transformations? $\diamondsuit$

1.2.2 Restricted Lorentz group:

If we are interested in two observers who who differ *only* in their state of motion then we want L to depend only on relative velocities, and possibly some rotation angles, and it had better be continuous as a function of relative velocity. Furthermore when the relative velocity is zero we want $L \to I$ [or at worst $L \to$ (rotation)]. So the "interesting" transformation matrices are those which are continuously connected to the identity, and in particular satisfy $\det(L) = +1$.

Theorem 2 *The set $\{L\}$ of matrices corresponding to the coordinate change between two observers who differ only in their state of motion is completely characterized by*

$$L^T\, \eta\, L = \eta; \qquad L \to I \text{ as } v \to 0.$$

This condition defines a group, in fact

$$\{L\} = \text{connected component of } [SO(3,1)].$$

$SO(3,1)$ denotes the group of simple pseudo-orthogonal matrices of signature $- + ++$. (The word "simple" means everything has determinant 1.) It is quite common to suppress the phrase "connected component of" and just talk about $SO(3,1)$.

1.2.3 Poincare group:

The Poincare group, as opposed to the Lorentz group, also includes the spacetime translations a.

$$P(X \to X') : X' = L\, X + a$$

Mathematically the Poincare group is the so-called "semi-direct product" of the translation group with the Lorentz group.

Notation: (Only used in the mathematics community; extremely rare or non-existent in the physics community.) The semi-direct product of a "base" group $(H, \cdot)$ with a second "transformation" group $(G, *)$ is defined only when G and H are both groups *and* G has a natural action on H: There must be a distinguished mapping from $G \times H \to H$. In that case the semidirect product is typically denoted

$$H \rtimes G$$

or less commonly by

$$H : G$$

The semidirect product is itself a group, with elements $(g, h) \in G \times H$, and with the semi-direct group multiplication defined by

$$(g, h) \circ (g', h') = (g * g', [g'\, h] \cdot h')$$

♢

That is, in our particular case:

$$(\text{Poincare group}) = (\text{Translation group}) \rtimes (\text{Lorentz group})$$

Exercise: Define the Poincare transformations as the collection of ordered pairs $P = (L, a)$ where L is a Lorentz transformation and a is a 4-vector. We are to think of the Poincare transformations as acting on spacetime position according to the rule

$$X \to P(X) = LX + a$$

Identify a physically natural way of defining a "multiplication/ composition" operation on Poincare transformations and verify that the Poincare transformations form a group under this operation. ♢

Exercise: Now verify that your physically natural definition of the "multiplication/ composition" operation on Poincare transformations not only makes the Poincare transformations a group — it also makes the Poincare transformations a semi-group in the sense defined above. ♢

1.2.4 Explicit Lorentz transformations

Let us now deduce the explicit form of the Lorentz transformations. First, there is one obvious subgroup of the Lorentz group, the rotation group. Indeed let R be any 3×3 orthogonal matrix and consider

$$L(R) = 1 \oplus R = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$$

This is a 4×4 matrix which explicitly satisfies the defining condition of the Lorentz group, but does not do anything to the time coordinate; it simply rotates x, y, z, among themselves.[6] (The two observers are stationary with respect to each other; rotated but not in relative motion.)

[6]Remember that for any orthogonal transformation (rotation) we have

$$R^T R = I; \qquad \text{that is} \qquad R^T = R^{-1}.$$

This lets us simplify the situation where the two observers are taken to be in relative motion: for any pair of observers we can without loss of generality take motion to be along the x axis since we can always rotate the coordinate systems to make it so.

Define $(1+1)$-dimensional matrices:

$$\eta_2 \equiv \begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix}; \qquad I_2 \equiv \begin{bmatrix} +1 & 0 \\ 0 & +1 \end{bmatrix}; \qquad \eta = \eta_2 \oplus I_2$$

Now for motion in the x direction we do not expect the y and z directions to be affected by the Lorentz transformation. That is we expect

$$L = L_2 \oplus I_2$$

Exercise: Verify, from the Einstein relativity principle, that this is in fact what happens. The easy part is to check that matrices of the form $L_2 \oplus I_2$ are Lorentz transforms in the $(3+1)$ dimensional sense. It is more subtle to show that when you look at the two spatial directions perpendicular to the direction of motion, the $(3+1)$ dimensional Lorentz transformations block diagonalize, and that on that perpendicular sub-block the $(3+1)$ dimensional Lorentz transformations reduce to the identity.

Hint: Do this in stages. First try to convince yourself that the Lorentz transformations block diagonalize. Then use the Einstein relativity principle to argue that the transformations in directions perpendicular to the direction of motion must be trivial. ♢

Assuming the results of this exercise, the defining condition for Lorentz transformations along the x axis now simplifies to[7]

$$L_2^T \, \eta_2 \, L_2 = \eta_2.$$

This implies

$$L_2^{-1} = \eta_2 \, L_2^T \, \eta_2.$$

Since this is a 2×2 matrix equation it is relatively easy to solve explicitly. Write

$$L_2 \equiv \begin{bmatrix} +a & +b \\ +c & +d \end{bmatrix}$$

Then using the fact $\det(L_2) = +1$, the explicit formula for L_2^{-1} coming from Cramer's rule for the matrix inverse is

$$L_2^{-1} = \begin{bmatrix} +d & -b \\ -c & +a \end{bmatrix}$$

[7] These "pure" Lorentz transformations, that by construction do not involve any rotation, are referred to as "boosts".

Explicitly multiplying out the matrices $\eta_2\, L_2^T\, \eta_2$, we have:

$$\eta_2\, L_2^T\, \eta_2 = \begin{bmatrix} +a & -c \\ -b & +d \end{bmatrix}$$

Therefore

$$\begin{bmatrix} +d & -b \\ -c & +a \end{bmatrix} = \begin{bmatrix} +a & -c \\ -b & +d \end{bmatrix}$$

So the four components are subject to three independent constraints, $a = d$, $b = c$, and and $\det(L_2) = +1$. The general solution is

$$L_2 \equiv \frac{1}{\sqrt{1-\beta^2}} \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix}$$

At this stage β is some arbitrary real number, physical interpretation not yet specified.

Exercise: Verify this is a solution. Verify that it is the only solution. $\diamondsuit$

Exercise: Verify that $L_2(-\beta) = L_2(\beta)^{-1}$. $\diamondsuit$

It is algebraically convenient to now define $\gamma = 1/\sqrt{1-\beta^2}$. (As yet, the physical interpretation of the parameters β and γ is unspecified.)

Exercise: Verify that any arbitrary Lorentz transformation can always be put in the form

$$L = BR$$

where B is a pure boost (in some direction) and R is a pure rotation. $\diamondsuit$

Exercise: Verify that any arbitrary Lorentz transformation can always be put in the form

$$L = R_1 \left[\begin{array}{cc|cc} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] R_2$$

Here R_1 and R_2 are pure rotation matrices, and the matrix in the middle is a pure boost in the x direction. $\diamondsuit$

Then we can write the Lorentz transformations as

$$ct' = \gamma\, (ct - \beta x);$$

$$x' = \gamma\, (x - \beta ct);$$

$$y' = y;$$

$$z' = z.$$

Now let's physically interpret β. Consider an object that according to the first observer is at rest at the origin. Then its path through spacetime, its world-line, is

$$X(t) = (ct; 0, 0, 0).$$

As seen by the second observer its path is

$$X'(t) = (ct'(t); \vec{x}'(t)) = (\gamma ct; -\gamma\beta ct, 0, 0)$$

So according to the second observer this object is moving with velocity

$$v_{12} = \frac{\Delta x'}{\Delta t'} = \frac{-\gamma\beta ct}{\gamma t} = -\beta c$$

That is: an object at rest with respect to the first observer is moving with velocity $v_{12} = -\beta c$ with respect to the second observer. Equivalently, the second observer is moving with velocity $v_{21} = +\beta c$ with respect to the first observer. This gives us the physical interpretation

$$\beta = v/c.$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

And now:

$$t' = \gamma \left(t - \frac{vx}{c^2}\right);$$

$$x' = \gamma (x - vt);$$

$$y' = y;$$

$$z' = z.$$

These are the Lorentz transformations (for a relative velocity v in the x direction).

Suppose we take the (formal and unphysical) limit $c \to \infty$, keeping v fixed. Then $\gamma \to 1$ and

$$t' \to t;$$

$$x' \to x - vt;$$

$$y' = y;$$

$$z' = z.$$

These are Galileo's transformations of Galilean relativity, the foundations of Newtonian mechanics. For almost all everyday life situations this is quite sufficient.

Exercise: Take the full Lorentz transformation and assume $v \ll c$, that is $\beta = v/c \ll 1$ so that $\gamma = 1/\sqrt{1-\beta^2} = 1$. Then the Lorentz transformations reduce to

$$t' \approx t - \frac{vx}{c^2};$$

$$x' \approx x - vt;$$

$$y' = y;$$

$$z' = z.$$

These are not quite Galileo's transformations. What is that vx/c^2 term doing there? Carefully analyze the situations under which that term can safely be neglected. (It's *not* as simple as just saying $v \ll c$.) $\diamondsuit$

Comment: So far I have done everything in "index-free" notation, emphasizing that Lorentz transformations are simply matrices that act on vectors consisting of 4 real numbers. One reason for doing this is that no-one seriously doubts the internal consistency of matrix algebra and vector spaces [8] — and in the approach I've adopted so far questions of the internal consistency of special relativity are simply seen to be questions regarding the internal consistency of matrix algebra. $\diamondsuit$

Warning: The question of whether or not a physical theory is internally consistent is a mathematical one that can be settled by pure thought without any appeal to experiment. In contrast, the question of whether or not the theory is an adequate representation of empirical reality is a logically separate question that can *only* be answered by experiment. $\diamondsuit$

Warning: Most of the salient features of Einstein's special relativity can adequately be treated by using mathematical techniques from elementary high-school algebra. Unfortunately this means that anyone capable of getting elementary high-school algebra wrong thinks they've made a new and exciting fundamental discovery. $\diamondsuit$

Exercise: Now suppose we perform a Lorentz transformation in some arbitrary direction $\hat{n}$ (instead of along the x axis). That is $\vec{v} = v\,\hat{n} = \beta\,c\,\hat{n}$. Show that in this case:

$$t \to t' = \gamma\left(t - \frac{\vec{v}\cdot\vec{x}}{c^2}\right);$$

$$\vec{x} \to \vec{x}' = [I - \hat{n}\otimes\hat{n}]\,\vec{x} + \gamma(\vec{x}\cdot\hat{n} - vt)\,\hat{n}.$$

[8] Well, hardly anyone seriously doubts the internal consistency of matrix algebra and vector spaces. However, for any whacked out piece of psychotic drivel you can imagine there will exist at least one proponent on the internet — the proof is left as an exercise for your favourite search engine.

These are the general Lorentz transformations.

Warning: There are *many* different ways of getting to these general Lorentz transformations, of which I have only presented *one*. If you do not like the particular way I have done things, feel free to rearrange the discussion to your taste — but you had better get to the same equations at the end of it all. $\diamondsuit$

Exercise: Consider the "quasi-Lorentz transformations"

$$t \to t' = \Omega(v)\ \gamma \left(t - \frac{\vec{v} \cdot \vec{x}}{c^2} \right);$$

$$\vec{x} \to \vec{x}' = \Omega(v)\ \left\{ [I - \hat{n} \otimes \hat{n}]\ \vec{x} + \gamma(\vec{x} \cdot \hat{n} - vt)\ \hat{n} \right\}.$$

where $\Omega(v)$ is some function of v. Explicitly show that these "quasi-Lorentz transformations" preserve the speed of light, but that they do not respect the Einstein relativity principle. $\diamondsuit$

1.3 Quaternions (an aside)

[Ignore this subsection for study purposes — it's here only if you are interested.]

Quaternions are a generalization of the complex numbers that were discovered by Sir William Rowan Hamilton (as in Hamiltonian mechanix) in the mid 1800's, and are particularly useful in dealing with 3-dimensional rotations.

Quaternions are so nice mathematically that they look as though they should be useful for special relativity as well... (But no one has ever made them quite work in any *clean* and *compelling* manner.)

Complex numbers can always be written as a linear combination of two real numbers:

$$c = r_0 + i r_1;$$

where the complex unit "i" is uniquely defined by the multiplication rule:

$$i^2 = -1.$$

In a similar way the quaternions are defined as linear combinations of four real numbers:

$$q = r_0 + \mathbf{i}\ r_1 + \mathbf{j}\ r_2 + \mathbf{k}\ r_3,$$

where there are now three complex units "**i**", "**j**", and "**k**", which are uniquely defined by the multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1;$$

and:

$$\mathbf{i}\,\mathbf{j} = -\mathbf{j}\,\mathbf{i} = \mathbf{k}.$$

Note that this multiplication law is not commutative.

Exercise: Prove:

1. The symmetric relationships:

$$\mathbf{i}\,\mathbf{j} = \mathbf{k}; \qquad \mathbf{j}\,\mathbf{k} = \mathbf{i}; \qquad \mathbf{k}\,\mathbf{i} = \mathbf{j}.$$

2. The triple law:

$$\mathbf{i}\,\mathbf{j}\,\mathbf{k} = -1.$$

3. Calculate the commutators $[a, b] = ab - ba$:

$$[\mathbf{i}, \mathbf{j}] = ???; \qquad [\mathbf{j}, \mathbf{k}] = ???; \qquad [\mathbf{k}, \mathbf{i}] = ???.$$

   Do they remind you of anything?

4. Calculate the anti-commutators $\{a, b\} = ab + ba$:

$$\{\mathbf{i}, \mathbf{j}\} = ???; \qquad \{\mathbf{j}, \mathbf{k}\} = ???; \qquad \{\mathbf{k}, \mathbf{i}\} = ???.$$

Interpret your results. ♢

Exercise:

1. Now define the quaternion-valued vector vector $\vec{\mathbf{e}} = (\mathbf{i}, \mathbf{j}, \mathbf{k})$ so that $\vec{\mathbf{e}}$ is a "vector" containing the three elements "$\mathbf{i}$", "$\mathbf{j}$", and, "$\mathbf{k}$".

2. Also define ordinary three-vectors vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, these are simply vectors containing three real numbers.

3. By doing this we can write arbitrary quaternions "a" and "b" in the form:

$$a = a_0 + \mathbf{i}\,a_1 + \mathbf{j}\,a_2 + \mathbf{k}\,a_3 = a_0 + \vec{\mathbf{e}} \cdot \vec{a};$$

$$b = b_0 + \mathbf{i}\,b_1 + \mathbf{j}\,b_2 + \mathbf{k}\,b_3 = b_0 + \vec{\mathbf{e}} \cdot \vec{b};$$

   where $\vec{\mathbf{e}} \cdot \vec{a}$ and $\vec{\mathbf{e}} \cdot \vec{b}$ are ordinary vector inner products ("dot products").

4. Calculate:

$$a^2 = ???;$$
$$b^2 = ???;$$
$$ab = ???;$$
$$ba = ???;$$
$$[a, b] = ???;$$
$$\{a, b\} = ???.$$

5. Hint: Verify

$$a\, b = (a_0\, b_0 - \vec{a} \cdot \vec{b}) + \vec{\mathbf{e}} \cdot (a_0\, \vec{b} + b_0\, \vec{a} + \vec{a} \times \vec{b}).$$

Interpret your results. ♢

All of the classic theory of rotations in 3 dimensions can *easily* and *cleanly* be rephrased in terms of quaternions; and it *seems* that you should be able to re-phrase special relativity in terms of quaternions as well.

Exercise: Define the "real part" of a quaternion in the obvious manner

$$\mathrm{Re}(q) = \mathrm{Re}(q_0 + \mathbf{i}\, q_1 + \mathbf{j}\, q_2 + \mathbf{k}\, q_3) = q_0.$$

Calculate:

$$\mathrm{Re}(ab) = ???.$$

Interpret your result. ♢

Exercise: Do a little digging into the literature (and on the internet) to see what else can be done with quaternions. ♢

Exercise: Try to understand why quaternions have become so important in modern computer graphics.

(It is important here to distinguish abstract mathematical isomorphism from practical questions of data storage and manipulation.) ♢

Research problem: Find a *clean* and *useful* way of representing all special relativity, including the Lorentz transformations, in terms of these real-coefficient quaternions.

[This is nowhere near as easy as it first looks; that's why I specified the qualifying adjectives *clean* and *useful*. This is not a homework exercise but any ideas you have will be looked at carefully and greatly encouraged.] ♢

1.4 Bi-quaternions (an aside)

[Ignore this subsection for study purposes — it's here only if you are interested.]

Another very useful concept is that of the bi-quaternion (complexified quaternion)

$$Z = z_0 + \mathbf{i}\, z_1 + \mathbf{j}\, z_2 + \mathbf{k}\, z_3,$$

where the z_a $(a = 0 \dots 3)$ are now *complex* numbers. (And so we really do need to distinguish the i occurring inside each z_a from the quaternionic basis element $\mathbf{i}$.) You then need to distinguish at least *three* types of complex conjugate:

$$\bar{Z} = \bar{z}_0 + \mathbf{i}\, \bar{z}_1 + \mathbf{j}\, \bar{z}_2 + \mathbf{k}\, \bar{z}_3;$$

$$Z^* = z_0 - \mathbf{i}\, z_1 - \mathbf{j}\, z_2 - \mathbf{k}\, z_3;$$

$$Z^\dagger = \bar{z}_0 - \mathbf{i}\, \bar{z}_1 - \mathbf{j}\, \bar{z}_2 - \mathbf{k}\, \bar{z}_3.$$

These ideas go back (at least) to Ludwik Silberstein in 1912.

- Ludwik Silberstein (1912), "Quaternionic form of relativity", Philosophy Magazine, Series 6 23: 790–809.
- Ludwik Silberstein (1914), The Theory of Relativity. (Macmillan)

Exercise: Can you use this formalism to make the previous exercises for real-coefficient quaternions simpler? ♢

Exercise: Show that there is an isomorphism

$$(i\mathbf{i}, i\mathbf{j}, i\mathbf{k}) \sim (\sigma_x, \sigma_y, \sigma_z)$$

between the "imaginary quaternions" and the Pauli sigma matrices. ♢

Project: Find a *clean* and *useful* way of representing Maxwell's electromagnetism in terms of bi-quaternions. ♢

Research problem: Find a *clean* and *useful* way of representing all special relativity, including the Lorentz transformations, in terms of bi-quaternions.

[This is nowhere near as easy as it first looks; that's why I specified the qualifying adjectives *clean* and *useful*. Start by looking up "biquaternions" on the web. This is not a homework exercise but any ideas you have will be looked at carefully and greatly encouraged.] ♢

1.5 Causal structure

Given any two events 1 and 2, we define $\Delta X \equiv X_2 - X_1$.

By construction the quantity $\eta(\Delta X, \Delta X)$ is a Lorentz invariant.

Definition 2
— *If* $\eta(\Delta X, \Delta X) < 0$ *we say the two events are timelike separated.*
— *If* $\eta(\Delta X, \Delta X) = 0$ *we say the two events are lightlike separated (null separated).*
— *If* $\eta(\Delta X, \Delta X) > 0$ *we say the two events are spacelike separated.*

- By definition, if the two events are timelike separated then $c|\Delta t| > |\Delta \vec{x}|$, so to travel from one event to the other you would need to travel at a speed $|\Delta \vec{x}|/|\Delta t| < c$. This means that normal subluminal particles (and so implicitly observers and/or their entire laboratories) can successfully travel from one event to the other.
- By definition, if the two events are lightlike separated then $c|\Delta t| = |\Delta \vec{x}|$, so to travel from one event to the other you would need to travel at exactly the speed $|\Delta \vec{x}|/|\Delta t| = c$.
  (So a photon could get from one event to the other, but no normal laboratory-based observer could possibly do so.)
- By definition, if the two events are spacelike separated then $c|\Delta t| < |\Delta \vec{x}|$, so to travel from one event to the other you would need to travel at a speed $|\Delta \vec{x}|/|\Delta t| > c$. That is, you would have to travel faster than light to get from one event to the other.

Faster than light travel is (in general) strongly discouraged: The basic issue is one of causality. In Newtonian physics if $\Delta t > 0$ for one observer, it is positive for all observers [in fact it is the *same* for all observers]. So everyone agrees on which of two events happens first. If they are simultaneous, $\Delta t = 0$, everyone agrees on this. This is the key feature of causality in Newtonian physics that does not carry over to special relativity.

Suppose we have two events that are spacelike separated, then under a Lorentz transformation

$$\Delta t' = \gamma \, \Delta t \, \left\{ 1 - \frac{v}{c^2} \frac{\Delta x}{\Delta t} \right\}.$$

Define

$$v_{critical} \equiv c^2 \left(\frac{\Delta x}{\Delta t} \right)^{-1}.$$

For spacelike separated events $|\Delta x / \Delta t| > c$ so $|v_{critical}| < c$.

If we now pick $v > v_{critical}$ (both velocities positive) or $v < v_{critical}$ (both velocities negative) then we can make Δt *change sign* by performing a Lorentz transformation.

That means that different observers will disagree on which event happens first, and so there is no generic way of defining a "total time ordering" for spacelike separated events in special relativity.

The best you can do is to define a pair of Lorentz invariant *partial orderings*:

Definition 3
An event X_1 chronologically precedes an event X_2, denoted $X_1 < X_2$ if both

$$\eta(\Delta X, \Delta X) < 0$$

and

$$\Delta t > 0.$$

Definition 4
An event X_1 causally precedes an event X_2, denoted $X_1 \ll X_2$ if both

$$\eta(\Delta X, \Delta X) \leq 0$$

and

$$\Delta t \geq 0.$$

Exercise:

- Look up the technical definition of a partial ordering.
- Verify that the definitions above (chronologically precedes, causally precedes) satisfy the axioms of a partial ordering.

  (You will need to distinguish a "strict partial order" from a "non-strict partial order".)

  (Note that we have discussed/will discuss partial orders in Math464 [differential geometry] when dealing with elementary notions of topology.)
- Verify that these definitions are Lorentz invariant, so that any two observers will agree on chronological precedence and causal precedence.

$\diamondsuit$

This is also why FTL (faster than light) is bad: If you could travel faster than light then two points on your world-line would be spacelike separated, and two different observers could disagree on the time order — this means that for an object travelling faster than light there is no sensible way to define which direction along the world-lime is "older" and which is "younger". Worse, without additional structure that goes far beyond that of special relativity, (and is violently in conflict with the underlying ideas of special relativity), it is then possible to set up closed causal loops (closed timelike curves or closed null curves) so that you can send a message to yourself five minutes ago. This is very bad. In special relativity the standard way out of this is to baldly say "you cannot travel faster than light".

(A more precise statement is that if you could travel faster than light you would need to severely modify special relativity to prevent closed causal curves.)

In the context of general relativity Stephen Hawking has promulgated a related idea: his "chronology protection conjecture" (which might be better thought of as a "chronology protection principle"). This is an example of modifying the rules of classical general relativity to make the theory better behaved under extreme conditions.

We can define the chronological (and causal) past (and future) of a point by using the partial orderings of chronological precedence and causal precedence.

Definition 5 (Chronological past and future)

$$J^-(X) = \{Y : Y < X\}; \qquad J^+(X) = \{Y : X < Y\}.$$

Definition 6 (Causal past and future)

$$I^-(X) = \{Y : Y \ll X\}; \qquad I^+(X) = \{Y : X \ll Y\}.$$

Note that the usual topology one places on Minkowski spacetime is just the standard one on $I\!R^4$. You can define an alternative topology by defining so-called "causal diamonds"[9]

$$B(X,Y) = J^+(X) \cap J^-(Y)$$

and taking these to be a sub-basis for the new topology. The resulting variant topology is called the **Alexandrov topology**[10] (or "causal topology") is *not* standard, but has been used for instance by Roger Penrose in his booklet "Techniques of differential topology in Relativity" as a way of studying causal structure in curved spacetime.

1.6 Lorentzian metric

You can use the distinguished quadratic form to define a Lorentzian metric.

[9]Strictly speaking, causal diamonds are diamond-shaped only in 1+1 dimensions. The terminology is unfortunately standard.

[10]Unfortunately, usage is not entirely consistent, and the phrase "Alexandrov topology" is sometimes also used in other situations as well.

Definition 7 (Version 1)
Let

$$d(X, Y) = \sqrt{\eta([X - Y], [X - Y])} = \sqrt{(X - Y)^T \, \eta \, (X - Y)}.$$

By construction this is Lorentz invariant. It is either real positive (spacelike separation), zero (lightlike separation), or pure imaginary (timelike separation). In particular this version of the Lorentzian metric is not positive semi-definite, and not even real. It does not in general satisfy the triangle inequality. Nevertheless, it has enough similarities to a metric that it is conventional to call it a Lorentzian metric (or pseudo-Riemannian metric).

Warning: Usage is not entirely consistent here; even within the physics community. $\diamondsuit$

Definition 8 (Version 2)
Let

$$d(X, Y) = \sqrt{|\eta([X - Y], [X - Y])|} = \sqrt{|\,(X - Y)^T \, \eta \, (X - Y)\,|}.$$

By construction this is Lorentz invariant. This version of the Lorentzian metric is *always real and positive semi-definite, though it does not in general satisfy the triangle inequality. Nevertheless, it has enough similarities to a metric that it is conventional to call it a Lorentzian metric (or pseudo-Riemannian metric).*

Warning: Physicists generally drop the qualifier "Lorentzian" or "pseudo-Riemannian" and just call this a metric. This has the potential to seriously confuse the mathematicians. Unfortunately the usage is now firmly established and it would be well nigh impossible to change it. $\diamondsuit$

The definitions most commonly in use in the mathematics community are slightly different. First a reminder:

Definition 9 (Metric — mathematics standard definition)
A metric function satisfies

$$d(x, y) \geq 0.$$

$$d(x, y) = 0 \iff x = y.$$

$$d(x, y) = d(y, x).$$

$$d(x, z) \leq d(x, y) + d(y, z).$$

More tricky is the mathematicians' definition of pseudo-metric and quasi-metic:

Definition 10 (Pseudo-metric — mathematics standard definition)
A pseudo-metric (sometimes called a semi-metric) satisfies

$$d(x,y) \geq 0.$$

$$d(x,x) = 0.$$

$$d(x,y) = d(y,x).$$

$$d(x,z) \leq d(x,y) + d(y,z).$$

So a pseudo-metric is allowed to have distinct points that are separated by zero distance. But this particular *definition of pseudo-metric still enforces the triangle inequality.*

Definition 11 (Quasi-metric — mathematics standard definition)
A quasi-metric satisfies

$$d(x,y) \geq 0.$$

$$d(x,x) = 0.$$

$$d(x,y) = 0 = d(y,x) \Rightarrow x = y$$

$$d(x,z) \leq d(x,y) + d(y,z).$$

So a quasi-metric is allowed to be non-symmetric.

Exercise: By looking at equivalence classes under the zero distance condition, show that we can always turn a pseudo-metric (in the mathematician's sense above) into a metric on the equivalence classes. $\diamondsuit$

Exercise: Show that a Lorentzian metric is not a pseudo-metric in the mathematician's sense above, because a Lorentzian metric does not in general satisfy the triangle inequality. (A single counter-example is sufficient; and you could simplify life by only considering the t–x plane.) $\diamondsuit$

Exercise: Show that if we start with a quasi-metric $d(x,y)$ and define

$$\tilde{d}(x,y) = \frac{d(x,y) + d(y,x)}{2}$$

then $\tilde{d}(x,y)$ is a metric. $\diamondsuit$

Now consider what happens if you pick some fixed observer, and restrict attention to his/her constant time slice — then the Lorentzian metric really is a metric on that time slice. To formalize this, consider:

Definition 12 (Spacelike hyperplane)
A 3-dimensional plane in Minkowski space is said to be a spacelike hyperplane iff for any two points X and Y on the plane the difference $X - Y$ is spacelike.

Exercise: Show that on any spacelike hyperplane a Lorentzian metric in the full Minkowski space induces a true metric on the spacelike hyperplane. $\diamondsuit$

Definition 13 (Timelike hyperplane)
A 3-dimensional plane in Minkowski space is said to be a timelike hyperplane iff there exist at least two points X and Y on the plane such that the difference $X - Y$ is timelike.

Exercise: Show that on any timelike hypersurface there will also be pairs of points that are spacelike and lightlike separated. $\diamondsuit$

Definition 14 (Lightlike [null] hyperplane)
A 3-dimensional plane in Minkowski space is said to be a lightlike [null] hyperplane iff there exist at least two points X and Y on the plane such that the difference $X - Y$ is lightlike, and if no two points on the plane are timelike separated.

Exercise: Show that on any lightlike hypersurface there will also be pairs of points that are spacelike separated. $\diamondsuit$

Again recall that the usual topology on Minkowski spacetime is just the standard one on $\mathbb{R}^4$. You can define yet another topology by defining

$$B(X, \delta) = \{Y : |d(X, Y)| < \delta\}$$

and taking these to be the basis for a non-standard topology. The resulting **variant** of the **Alexandrov topology** is again *not* standard, and physicists (almost) never use it.[11]

Exercise: Explore the relationship between this Alexandrov topology (defined in terms of the Lorentzian metric) and the previously discussed Alexandrov topology (defined in terms of the causal structure). $\diamondsuit$

Exercise: Suppose the three events X, Y, and Z are all pairwise spacelike separated from each other. Prove that the triangle inequality is *sometimes* not satisfied. $\diamondsuit$

[11] Usage is not entirely consistent, and the phrase "Alexandrov topology" is sometimes also used in other situations.

Exercise: Suppose three events X, Y, and Z uniquely specify a two-dimensional spacelike plane. Show that these three events are all pairwise spacelike separated from each other. Prove that the triangle inequality is satisfied. $\diamondsuit$

Exercise: Suppose three events X, Y, and Z uniquely specify a two-dimensional timelike plane. What can you say about the triangle inequality? $\diamondsuit$

Exercise: Suppose three events X, Y, and Z uniquely specify a two-dimensional lightlike plane. What can you say about the triangle inequality? $\diamondsuit$

Exercise: Suppose the three events X, Y, and Z are all pairwise timelike separated from each other. Prove that

$$|d(X,Y)| \geq |d(X,Z)| + |d(Z,Y)|.$$

That is, in this situation the triangle inequality is maximally *violated*. (Indeed we have an *anti-triangle inequality*).

Physically interpret this result in terms of the "twin pseudo-paradox". $\diamondsuit$

Exercise: Suppose X and Z are *any* two points in Minkowski space. Show that $\forall \epsilon > 0$ it is possible to pick a point Y in such a way that

$$d(X,Y) + d(Y,Z) < \epsilon.$$

This implies that *any* two points X and Z in Minkowski space can be connected by an arbitrarily short curve.

Note that this typically will *not* be a straight line. $\diamondsuit$

Now suppose we consider the "complex" version of the distance function,

$$d(X,Y) = \sqrt{\eta([X-Y],[X-Y])}.$$

Now square both sides

$$d(X,Y)^2 = \eta([X-Y],[X-Y]) = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta x^2$$

This combination is by construction a Lorentz invariant and is called the "invariant interval". It is often written as

$$\Delta s^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta x^2$$

If we are considering "infinitesimal" displacements this is very commonly written as

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

This "infinitesimal" form has the great advantage that it can naturally be adapted to the curved spacetimes of the general relativity.

Definition 15 (Invariant interval)

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta x^2$$

"Infinitesimal" version:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

1.7 Inner product

Now suppose you have somehow constructed two quantities A and B that transform the same way as spacetime position. We call these quantities 4-vectors.

Definition 16 *For any two 4-vectors A and B the inner product*

$$\eta(A, B)$$

is a Lorentz invariant.

Exercise: Check this. ♢

Exercise: Find explicit coordinate formulae for

$$\eta(A, B) = ???$$

in terms of the components of A and B. ♢

Definition 17 *Two 4-vectors A and B are said to be orthogonal [4-orthogonal] if the inner product*

$$\eta(A, B) = 0.$$

Exercise: For a spacelike hypersurface, define the notion of a normal vector in terms of a 4-vector that is 4-orthogonal to every displacement in the hypersurface. Show that this normal vector is timelike. ♢

Exercise: For a timelike hypersurface, define an appropriate notion of normal vector and demonstrate that this normal vector is spacelike. ♢

Exercise: Show that every lightlike vector is orthogonal to itself. ♢

Exercise: Show that for a lightlike hypersurface the normal to the hypersurface lies in the plane of the hypersurface. ♢

1.8 Index-based methods for SR

It is now useful to start introducing some index-notation.

Comment: You may at this stage find it useful to scan over parts of the Math 464/Math465 notes. Maybe the section on coordinate charts and atlases. ♢

Put covariant indices on the matrix η so that

$$\eta_{ab} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}$$

and put a contravariant index on the 4-position so that

$$X^a = (ct; \vec{x}).$$

Then for consistency the Lorentz transformation should be written with one covariant index and one contravariant index so that

$$X^a \to \bar{X}^a = L^a{}_b \, X^b$$

and in particular

$$L^a{}_b = \frac{\partial \bar{X}^a}{\partial X^b}.$$

That is, the Lorentz transformation is simply a special case of the rules for changing tensor components. The thing that is different is that the coordinates systems in question are global, covering the entire manifold [Minkowski space], and that the matrices $L^a{}_b$ are carefully constructed to be constant position-independent matrices.

The theorem $L^T \eta L = \eta$ is now rephrased as

$$L^a{}_b \, L^c{}_d \, \eta_{ac} = \eta_{bd}$$

That is: special relativity can simply be viewed as the study of those coordinate transformations that leave the special distinguished tensor η_{ab} invariant.

The inner product of two 4-vectors $\eta(A, B)$ is now rephrased as

$$\eta(A, B) = \eta_{ab} \, A^a \, B^b = -A^0 \, B^0 + \sum_{i=1}^{3} A^i \, B^i.$$

The distance between 4-positions X and Y is now

$$d(X, Y) = \sqrt{|\eta_{ab} \, [X - Y]^a \, [X - Y]^b|} = \sqrt{|-c^2 \Delta t^2 + \Delta x^2|}.$$

1.9 Relativistic combination of velocities

Exercise: [**Relative speed**] Suppose we have two different observers, of 4-velocities

$$V_1^a = \gamma_1(1, v_1^i) \qquad \text{and} \qquad V_2^a = \gamma_2(1; v_2^i).$$

Show that their relative speed (the speed of one observer as seen by the other) is

$$v = \frac{\sqrt{(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2/c^2}}{1 - \vec{v}_1 \cdot \vec{v}_2/c^2}$$

Hint: Calculate the invariant quantity $\eta(V_1, V_2) = \eta_{ab} V_1^a V_2^b$ in two different suitably chosen reference frames. ♢

Exercise: [**Thomas precession/Wigner rotation**]

Read and understand appendix A on the relativistic combination of velocities, Thomas precession, and Wigner rotation. This is an example of non-trivial "advanced" special relativity. ♢

1.10 Relativistic kinematics

For any timelike world-line, not necessarily a straight line, we can define "proper time" along the world-line as

$$c\tau = \int \sqrt{-\eta(\mathrm{d}X, \mathrm{d}X)} = \int \sqrt{-\eta\left(\frac{\mathrm{d}X}{\mathrm{d}\lambda}, \frac{\mathrm{d}X}{\mathrm{d}\lambda}\right)} \; \mathrm{d}\lambda$$

For an object at rest this conveniently becomes $\tau \to t$, but the above is completely general.

Exercise: Check that in index based language this is equivalent to

$$c\tau = \int \sqrt{-\eta_{ab} \, \frac{\mathrm{d}X^a}{\mathrm{d}\lambda} \, \frac{\mathrm{d}X^b}{\mathrm{d}\lambda}} \; \mathrm{d}\lambda$$

Try to interpret this result in terms of differential geometry. ♢

Now parameterize the world line $X(\tau)$ by this invariant proper time τ, and define the 4-velocity by

$$V(\tau) = \frac{\mathrm{d}X}{\mathrm{d}\tau}$$

By construction this is a 4-vector that transforms the same way as $X(\tau)$ itself. Note that by definition

$$\eta(V, V) = -c^2$$

Note that at some stage I will get tired of explicitly writing c and will adopt units of measure so that $c = 1$; for the time being I will try to keep c, but don't be suprised if it suddenly vanishes...

Now suppose there is some quantity, denoted m_0 which we will call the mass (also called invariant mass, proper mass, rest mass) and for massive objects ($m_0 \neq 0$) define the 4-momentum

$$P = m_0 \, V = m_0 \, \frac{\mathrm{d}X}{\mathrm{d}\tau}$$

Then by the chain rule

$$P = m_0 \, \frac{\mathrm{d}t}{\mathrm{d}\tau} \, \left(c; \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \right)$$

Now let's define $\mathrm{d}\vec{x}/\mathrm{d}t = \beta \, c \, \hat{n}$, in which case we can calculate

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

and so

$$P = m_0 \, c \, \gamma \; (1; \beta \, \hat{n}) \, .$$

Define

$$P = \left(\frac{E}{c}; \, \vec{p} \right)$$

then as $\beta \to 0$

$$\vec{p} = m_0 c \, \gamma \beta \, \hat{n} \to m_0 \vec{v},$$

and, [using the binomial expansion $\gamma = (1 - \beta^2)^{-1/2} = 1 + \frac{1}{2}\beta^2 + O(\beta^4)$],

$$E = m_0 c^2 \gamma \to m_0 c^2 + \frac{1}{2} m_0 v^2 + \dots$$

So $\vec{p}$ has the right limiting behaviour to be thought of as 3-momentum, while E (apart from a constant offset $m_0 c^2$) has the right limiting behaviour to be thought of as kinetic energy. Since this constant offset is there even if the object is at rest it is often referred to as the "rest energy". We now take these definitions as primary and define the special relativistic energy-momentum by

$$P = \left(\frac{E}{c}; \, \vec{p} \right); \qquad E = m_0 c^2 \, \gamma; \qquad \vec{p} = m_0 c \gamma \beta \, \hat{n}.$$

Note that

$$\eta(P, P) = -m_0^2 c^2,$$

which is often written as

$$E^2 - p^2 c^2 = m_0^2 c^4.$$

Exercise: Verify that to fourth order in v

$$E = m_0 c^2 \gamma \to m_0 c^2 + \frac{1}{2} m_0 v^2 + \frac{3}{8} \frac{m_0 v^4}{c^2} + O(v^6).$$

$\diamondsuit$

Exercise: Perform a full Taylor series expansion

$$E(v) = \sum_{n=0}^{\infty} a_n \, v^n$$

Evaluate all of the coefficients a_n. What is the radius of convergence of this Taylor series? Interpret this result physically — where is the closest singularity in the complex plane? $\diamondsuit$

Exercise: Verify that

$$\lim_{\beta \to 1} \left\{ \frac{c \, p(\beta)}{E(\beta)} \right\} = 1.$$

$\diamondsuit$

Exercise: Verify that

$$v(E) = c \sqrt{1 - \frac{m_0^2 \, c^4}{E^2}}.$$

$\diamondsuit$

Exercise: Verify that

$$v(p) = c \, \frac{p}{\sqrt{m_0^2 \, c^2 + p^2}}.$$

◊

Exercise: Relativistic Kinematics:

- If $P = (E/c, p)$ is a lightlike vector, what can we say about the speed of the particle?
- If $P = (E/c, p)$ is a timelike vector, what can we say about the speed of the particle?
- If $P = (E/c, p)$ is a spacelike vector, what can we say about the speed of the particle?
- Suppose we have two particles with four-momentum P_1 and P_2:

  $$P_1 = (E_1/c, p_1); \qquad P_2 = (E_2/c, p_2).$$

  Calculate:

  $$\eta(P_1, P_2) = P_1^T \; \eta \; P_2 = \; ???$$

  Physically interpret this Lorentz invariant in two separate ways, first by going to the rest frame of particle 1; and second by going to the rest frame of particle 2.

- Prove that if P_1 is timelike and P_2 is timelike, then $P_1 + P_2$ is timelike.

  (This is the situation corresponding to adding the four-momenta of two slower-than-light particles.)

- Prove that if P_1 is timelike or lightlike (null), and P_2 is timelike or lightlike (null), then $P_1 + P_2$ is timelike or lightlike (null).

  (This is the situation corresponding to the four-momenta of two physical particles, either slower than light or travelling at exactly the speed of light, being added.)

- If $P_1 + P_2$ is lightlike, and P_1 and P_2 are both individually physical (timelike or lightlike) then this can only happen if there is a very special relationship between P_1 and P_2.

  What is this relationship? Prove it. Interpret the result.

- Prove that the photon cannot decay into two slower-than-light particles.

  (That is, use conservation of 4-momentum, plus the fact that the invariant mass m_0 of the photon is zero, to place limits on the outgoing 4-momenta of any hypothetical decay process.)

◊

1.11 Relativistic dynamics

We now define the notion of 4-force, and relate it to the rate of change of the 4-momentum by:

$$F = \frac{\mathrm{d}P}{\mathrm{d}\tau} = m_0\,\frac{\mathrm{d}V}{\mathrm{d}\tau} + \frac{\mathrm{d}m_0}{\mathrm{d}\tau}\,V$$

Then

$$\eta(F,V) = m_0\,\eta\left(\frac{\mathrm{d}V}{\mathrm{d}\tau}, V\right) + \frac{\mathrm{d}m_0}{\mathrm{d}\tau}\,\eta(V,V).$$

Therefore

$$\eta(F,V) = \frac{1}{2}\,m_0\,\frac{\mathrm{d}}{\mathrm{d}\tau}\eta(V,V) + \frac{\mathrm{d}m_0}{\mathrm{d}\tau}\,\eta(V,V),$$

so that

$$\eta(F,V) = -\frac{1}{2}m_0\frac{\mathrm{d}}{\mathrm{d}\tau}c^2 - \frac{\mathrm{d}m_0}{\mathrm{d}\tau}c^2.$$

But c is a constant! Therefore, finally,

$$\eta(F,V) = -\frac{\mathrm{d}m_0}{\mathrm{d}\tau}c^2 = -\frac{\mathrm{d}(m_0c^2)}{\mathrm{d}\tau}.$$

So the inner product between 4-force and 4-velocity governs the rate of change of the invariant mass:

$$\frac{\mathrm{d}m_0}{\mathrm{d}\tau} = -\frac{1}{c^2}\,\eta(F,V) = -\frac{1}{c^2}\,F^a\,V_a.$$

Note that the invariant mass of a composite object *is* allowed to change as energy flows into and out of the system.

Examples: Absorbing heat will make a body heavier. Conversely, when nuclei emit gamma rays they lose a little bit of mass. At a much more prosaic level, the invariant mass of a rocket definitely changes as it burns fuel and exhaust gasses are forced out the nozzle.

Example: In contrast, the invariant mass of individual elementary particles [such as electrons protons or neutrons] does not change under any applied forces.

We can now rearrange the force equation to yield

$$m_0\,\frac{\mathrm{d}V}{\mathrm{d}\tau} = F - \frac{\mathrm{d}m_0}{\mathrm{d}\tau}V = F + \frac{\eta(F,V)V}{c^2}$$

That is

$$m_0\,\frac{\mathrm{d}V}{\mathrm{d}\tau} = F + \frac{\eta(F,V)V}{c^2}$$

Exercise: Show that in component language this can be written as

$$m_0 \, \frac{\mathrm{d}V^a}{\mathrm{d}\tau} = \left[\delta^a{}_b + \frac{V^a \, V_b}{c^2} \right] F^b$$

So that (mass)×(4-acceleration) = (the projection of the 4-force onto the three-plane perpendicular to the 4-velocity). ♢

Exercise: Show that

$$P^a{}_b = \left[\delta^a{}_b + \frac{V^a \, V_b}{c^2} \right]$$

is a projection operator. That is, verify that

$$P^a{}_b \, P^b{}_c = P^a{}_c.$$

♢

Now pick a particular reference frame of fixed 4-velocity V_0. This is the 4-velocity of some specific "observer", *not* the 4-velocity of the particle. Then:

$$-\eta(P, V_0) = \text{“Energy of the particle in the rest frame of } V_0\text{”} = E(P, V_0)$$

So

$$\frac{\mathrm{d}E(P, V_0)}{\mathrm{d}\tau} = -\eta \left(\frac{\mathrm{d}P}{\mathrm{d}\tau}, V_0 \right) = -\eta(F, V_0).$$

That is $\eta(F, V_0)$ is the *power* dumped into the particle (rate of change of energy per unit of particle proper time) as seen by the observer with 4-velocity V_0 .

1.12 Electromagnetic fields and forces

One of the most ubiquitous of forces is the electromagnetic force on a charged particle. In ordinary Newtonian mechanics we have the Lorentz force law

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{F} = q \left[\vec{E} + \vec{v} \times \vec{B}/c \right]$$

so that the force depends on the electric field, the magnetic field, and the 3-velocity. Since the whole point of SR was to account for the peculiar properties of electromagnetism, and in particular the observed fact that the speed of light is a universal observer-independent constant, we had better be able to take this Newtonian result and develop a clean SR generalization.

[Note that with this convention electric and magnetic fields have the same physical dimension and are measured in the same units — in SI one typically uses Volts/metre.]

Table of selected elementary particle invariant rest masses.

Particle	Mass
Photon	$< 1 \times 10^{-24}$ MeV
$W^{\pm}$	80.399 ± 0.023 GeV
Z^0	91.1876 ± 0.0021 GeV
Electron	$0.510998910 \pm 0.000000013$ MeV
Muon	105.658367 ± 0.000004 MeV
Tau	1776.82 ± 0.16 MeV
Proton	938.272013 ± 0.000023 MeV
Neutron	939.565346 ± 0.000023 MeV

Table 1.1: Some key particle masses. (Source: pdg.lbl.gov as of 5 July 2011.)

Note that neutrino masses are somewhat ambiguous due to neutrino mixing, and that quark and gluon masses are somewhat ambiguous due to quantum-chromodyamic (QCD) confinement. I have only listed those elementary particles for which the invariant rest mass is well-defined in the sense of the existence of suitable asymptotic states.

Now we have already seen that in SR we can use the "proper time" to parameterize the world-line of a particle and so define 4-velocity and 4-acceleration [flat space at this stage]

$$V^a = \frac{\mathrm{d}X^a(\tau)}{\mathrm{d}\tau}; \qquad A^a = \frac{\mathrm{d}V^a(\tau)}{\mathrm{d}\tau}$$

Experimental fact: All known elementary particles have well defined rest masses. $\diamondsuit$

In particular when we fire an electron down a cathode ray tube, and measure its energy and momentum, we find that

$$E^2 - p^2c^2 = m_0^2 c^4$$

always give the same value of m_0 for all electrons regardless of what sort of electric or magnetic fields we have used to accelerate the electron (or proton, or whatever elementary particle we choose to play with). This tells us that when subject to electromagnetic forces

$$\eta(F_{em}, V) = 0.$$

That is, the electromagnetic force is always 4-orthogonal to the 4-velocity. If this were not the case then all those lovely tables of "rest mass" for various elementary particles simply would not exist.

The 4-orthogonality of the 4-force and 4-velocity is not enough to specify the force law completely, but does suggest that it might be worth looking at what we could do by using a 2-form F^{ab}. Consider a force law of the form:

$$F^a_{\text{anzatz}} = q\ F^{ab}\ V_b$$

then automatically $F^a_{\text{anzatz}}\ V_a = \eta(F_{\text{ansatz}}, V) = 0$, so that such a force law leaves the rest mass unchanging.

Warning: You will need to distinguish F^a the 4-force, from F^{ab} the field strength, by context. The notation is unfortunately standard. $\diamondsuit$

Furthermore this ansatz above would imply

$$\frac{\mathrm{d}P^a}{\mathrm{d}\tau} = q\ F^{ab}\ \frac{\mathrm{d}X_a}{\mathrm{d}\tau}$$

so that in terms of laboratory time derivatives (rather than proper-time derivatives)

$$\frac{\mathrm{d}P^a}{\mathrm{d}t} = q\ F^{ab}\ \frac{\mathrm{d}X_a}{\mathrm{d}t}.$$

Thus

$$\frac{\mathrm{d}P^0}{\mathrm{d}t} = q\ F^{0i}\ \frac{\mathrm{d}X^i}{\mathrm{d}t}$$

and

$$\frac{\mathrm{d}P^i}{\mathrm{d}t} = q\ \left[-F^{i0}\ c + F^{ij}\ \frac{\mathrm{d}X^j}{\mathrm{d}t}\right]$$

Now suppose we identify

$$F^{i0} \leftrightarrow -E^i/c; \qquad F^{ij} \leftrightarrow +\epsilon^{ijk} B_k/c$$

Then the second of these equations is simply the Lorentz force law,

$$\frac{\mathrm{d}P^i}{\mathrm{d}t} = q\ \left[E^i + (\vec{v}\times\vec{B})^i\right],$$

while the first is the statement that

$$(\text{power}) = \frac{\mathrm{d}(\text{energy})}{\mathrm{d}t} = q\ \vec{E}\cdot\vec{v}$$

Therefore we have identified a useful ansatz for the electromagnetic force:

$$F^a_{em} = q\ F^{ab}\ V_b; \qquad F^{ab} = \frac{1}{c}\left[\begin{array}{c|c} 0 & +\vec{E} \\ \hline -\vec{E} & *B \end{array}\right]$$

Here $*B$ is the 3-dimensional Hodge dual in exactly the sense some of you may have seen in Math 464. This force law gives you three things: an unchanging rest mass, the Lorentz force law, *and* the energy equation. It also gives you a natural way of looking at electric and magnetic fields as different parts of a 2-form; the electric field is the (space)-(time) portion while the magnetic field is simply the (space)-(space) portion.

Exercise: [Trivial] Interpret the (time)-(time) part of the 2-form F^{ab}. What is F^{00}? $\diamondsuit$

1.12.1 Lorentz transformations for electromagnetic fields

Another side-effect of this discussion is that we can now see how electric and magnetic fields transform under Lorentz transformations. Since $X^a \to \bar{X}^a = L^a{}_b\, X^b$ it is clear that for a T^2_0 Cartesian tensor such as F^{ab} we must have

$$F^{ab} \to \bar{F}^{ab} = L^a{}_c\, L^b{}_d\, F^{cd}$$

In matrix notation

$$X \to \bar{X} = L\, X; \qquad F \to \bar{F} = L\, F\, L^T.$$

Transformation along the x direction

Let's be explicit about this. For a Lorentz transformation in the x direction we can write

$$L^a{}_b = \left[\begin{array}{cc|cc} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

We can also write

$$F^{cd} = \frac{1}{c}\left[\begin{array}{cc|cc} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ \hline -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{array}\right]$$

Now block multiply[12]

$$
\left[\begin{array}{cc|cc}
0 & \bar{E}_x & \bar{E}_y & \bar{E}_z \\
-\bar{E}_x & 0 & \bar{B}_z & -\bar{B}_y \\
\hline
-\bar{E}_y & -\bar{B}_z & 0 & \bar{B}_x \\
-\bar{E}_z & \bar{B}_y & -\bar{B}_x & 0
\end{array}\right]
$$

$$
= \left[\begin{array}{cc|cc}
\gamma & -\gamma\beta & 0 & 0 \\
-\gamma\beta & \gamma & 0 & 0 \\
\hline
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{cc|cc}
0 & E_x & E_y & E_z \\
-E_x & 0 & B_z & -B_y \\
\hline
-E_y & -B_z & 0 & B_x \\
-E_z & B_y & -B_x & 0
\end{array}\right]
\left[\begin{array}{cc|cc}
\gamma & -\gamma\beta & 0 & 0 \\
-\gamma\beta & \gamma & 0 & 0 \\
\hline
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right]
$$

Looking at the four 2×2 blocks this implies:

$$
\begin{bmatrix} 0 & \bar{E}_x \\ -\bar{E}_x & 0 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & E_x \\ -E_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix}
$$

and

$$
\begin{bmatrix} \bar{E}_y & \bar{E}_z \\ \bar{B}_z & -\bar{B}_y \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} E_y & E_z \\ B_z & -B_y \end{bmatrix}
$$

and

$$
\begin{bmatrix} 0 & \bar{B}_x \\ -\bar{B}_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & B_x \\ -B_x & 0 \end{bmatrix}
$$

It is easy to verify that

$$
\begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix}
$$

To check this you could try brute force matrix multiplication, or some tricks with symmetries.

Therefore:

- By the first of these 2×2 matrix equations we have

$$
\bar{E}_x = E_x
$$

[12]Suppose we take a big (rectangular) matrix and break it apart into smaller (rectangular) blocks; these are called "partitioned matrices". Then you should easily be able to convince yourself that matrix multiplication satisfies the "block multiplication" rule:

$$
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BC & AF + BH \\ CE + DG & CF + DH \end{bmatrix}
$$

If necessary, look up a few reference books. This is simply a standard trick of linear algebra. Note that the order in which the matrix multiplications is carried out is important.

- Furthermore, by the third of these 2×2 matrix equations we have

$$\bar{B}_x = B_x$$

- That is, the values of the electric and magnetic fields *in the direction of motion* are unaffected by the Lorentz transformations.

- In the directions perpendicular to the direction of motion we have (by the second of these 2×2 matrix equations)

$$\begin{bmatrix} \bar{E}_y & \bar{E}_z \\ \bar{B}_z & -\bar{B}_y \end{bmatrix} = \gamma \begin{bmatrix} E_y - \beta B_z & E_z + \beta B_y \\ B_z - \beta E_y & -B_y - \beta E_z \end{bmatrix}$$

That is:

$$\left(\; \bar{E}_y, \quad \bar{E}_z \; \right) = \gamma \left(\; E_y - \beta B_z, \quad E_z + \beta B_y \; \right);$$
$$\left(\; \bar{B}_z, \quad \bar{B}_y \; \right) = \gamma \left(\; B_z - \beta E_y, \quad B_y + \beta E_z \; \right).$$

This (in principle) completely specifies the Lorentz transformation properties of the electromagnetic field.

Summary:

For Lorentz transformations in the x direction

$$\bar{E}_x = E_x$$
$$\bar{B}_x = B_x$$
$$\left(\; \bar{E}_y, \quad \bar{E}_z \; \right) = \gamma \left(\; E_y - \beta B_z, \quad E_z + \beta B_y \; \right);$$
$$\left(\; \bar{B}_z, \quad \bar{B}_y \; \right) = \gamma \left(\; B_z - \beta E_y, \quad B_y + \beta E_z \; \right).$$

To get a deeper grasp on things it's advisable to look at what happens as you Lorentz transform along a general direction.

Exercise: Show that

$$\begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \propto \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}$$

From this deduce

$$\begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} = \left\{ \det \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \right\}^2 \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}$$

Finally, from this deduce

$$\begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}$$

without resorting to brute force matrix multiplication. ◇

Transformation along a general direction

This is somewhat tedious and slightly painful, but is something you should see at least once in your lives, just to be sure it really works.

Now let's consider a general direction. Recall that for a Lorentz transformation in the x direction we can write

$$L^a{}_b = \left[\begin{array}{cc|cc} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

So in a general direction $\hat\beta$ we have

$$L^a{}_b = \left[\begin{array}{c|c} \gamma & -\gamma\beta\,\hat\beta_j \\ \hline -\gamma\beta\,\hat\beta^i & \delta^i{}_j + (\gamma-1)\hat\beta^i\hat\beta_j \end{array}\right] = \left[\begin{array}{c|c} \gamma & -\gamma\beta^i \\ \hline -\gamma\beta^j & L^i{}_j \end{array}\right]$$

Therefore:

$$\begin{aligned} E^i \to \bar E^i &= c\bar F^{0i} = c\,L^0{}_c\,L^i{}_d\,F^{cd} \\ &= c\left\{L^0{}_0\,L^i{}_0\,F^{00} + L^0{}_0\,L^i{}_j\,F^{0j} + L^0{}_j\,L^i{}_0\,F^{j0} + L^0{}_j\,L^i{}_k\,F^{jk}\right\} \\ &= 0 + \gamma\,L^i{}_j\,E^j - \gamma^2\beta^i(\vec\beta\vec E) + \gamma\,L^i{}_k\,(*B)^{jk}\,\beta_j \end{aligned}$$

This is enough to tell you that electric and magnetic fields get mixed together under a Lorentz transformation. To proceed we first use the fact that

$$\gamma\,L^i{}_j - \gamma^2\beta^2\,\hat\beta^i\hat\beta_j = \gamma\delta^i{}_j + (\gamma^2-\gamma-\gamma^2\beta^2)\,\hat\beta^i\hat\beta_j = \gamma\left(\delta^i{}_j - \hat\beta^i\hat\beta_j\right) + \hat\beta^i\hat\beta_j$$

and secondly use the fact that

$$(*B)^{jk} = \begin{bmatrix} 0 & B^z & -B^y \\ -B^z & 0 & B^x \\ B^y & -B^x & 0 \end{bmatrix};$$

to then write

$$\beta_j\,(*B)^{jk} = -(B^y\beta^z - B^z\beta^y, B^z\beta^x - B^x\beta^z, B^x\beta^y - B^y\beta^x) = (\vec\beta\times\vec B)^k.$$

We then obtain

$$E^i \to \bar E^i = \gamma\,L^i{}_j\,E^j - \gamma^2\beta^i(\vec\beta\vec E) + \gamma\,L^i{}_k\,(\vec\beta\times\vec B)^k$$

which further simplifies to

$$E^i \to \bar E^i = \gamma\,(\delta^i{}_j - \hat\beta^i\hat\beta_j)\,E^j + \hat\beta^i\hat\beta_j\,E^j + \gamma\,(\vec\beta\times\vec B)^i$$

or (finally)

$$E^i \to \bar{E}^i = \hat{\beta}^i \hat{\beta}_j \; E^j + \gamma \; \left\{ (\delta^i{}_j - \hat{\beta}^i \hat{\beta}_j) \; E^j + (\vec{\beta} \times \vec{B})^i \right\}.$$

This is the general formula for the transformation of an electric field from one observer to another in SR. Note that the component of the electric field in the direction of the transformation is unaffected.

You can find the same result, deduced in various different ways, in Jackson "Electrodynamics", or indeed in any serious textbook on electromagnetism.

- John David Jackson, Classical Electrodynamics, (Wiley, New York, 1975).

We can find the transformation law for a magnetic field in a completely analogous manner. (It's just that now we will have a lot of Hodge star operations and Levi-Civita tensors floating around in the middle of the computation — and no this is not the way that Jackson does it.) Start with

$$\begin{aligned}
B^i \to \bar{B}^i &= \frac{c}{2} \; \epsilon^{ijk} \; \bar{F}^{jk} = \frac{c}{2} \; \epsilon^{ijk} \; L^j{}_c \; L^l{}_d \; F^{cd} \\
&= \frac{c}{2} \; \epsilon^{ijk} \left\{ L^j{}_0 \; L^k{}_l \; F^{0l} + L^j{}_l \; L^k{}_0 \; F^{l0} + L^j{}_l \; L^k{}_m \; F^{lm} \right\} \\
&= \frac{c}{2} \; \epsilon^{ijk} \left\{ [L^j{}_0 \; L^k{}_l - L^j{}_l \; L^k{}_0] \; E^l/c + L^j{}_l \; L^k{}_m \; F^{lm} \right\} \\
&= \epsilon^{ijk} \; \left\{ L^j{}_0 \; L^k{}_l \; E^l \right\} + \frac{c}{2} \; \epsilon^{ijk} \; L^j{}_l \; L^k{}_m \; F^{lm} \\
&= \epsilon^{ijk} \; \left\{ -\gamma \beta^j \; L^k{}_l \; E^l \right\} + \frac{c}{2} \; \epsilon^{ijk} \; L^j{}_l \; L^k{}_m \; F^{lm} \\
&= -\gamma (\vec{\beta} \times [L^\bullet{}_\bullet \; \vec{E}])^i + \frac{c}{2} \; \epsilon^{ijk} \; L^j{}_l \; L^k{}_m \; F^{lm} \\
&= -\gamma (\vec{\beta} \times \vec{E})^i + \frac{c}{2} \; \epsilon^{ijk} \; L^j{}_l \; L^k{}_m \; F^{lm}
\end{aligned}$$

Now let's concentrate on that last term:

$$\begin{aligned}
\frac{c}{2} \; \epsilon^{ijk} \; L^j{}_l \; L^k{}_l \; F^{lm} &= \frac{1}{2} \; \epsilon^{ijk} \; L^j{}_l \; L^k{}_m \; \epsilon^{lmn} B^n \\
&= \left[\frac{1}{2} \; \epsilon^{ijk} \; (\delta^j{}_l - [\gamma - 1] \hat{\beta}^j \hat{\beta}_l) \; (\delta^k{}_m - [\gamma - 1] \hat{\beta}^k \hat{\beta}_m) \epsilon^{lmn} \right] B^n \\
&= \left[\frac{1}{2} \; \epsilon^{ijk} \; \left(\delta^j{}_l \; \delta^k{}_m - [\gamma - 1] \left(\hat{\beta}^j \hat{\beta}_l \; \delta^k{}_m + \hat{\beta}^k \hat{\beta}_m \; \delta^j{}_l \right) \right) \epsilon^{lmn} \right] B^n \\
&= \left[\frac{1}{2} \; \epsilon^{ijk} \; \epsilon^{jkn} + [\gamma - 1] \left(\epsilon^{ijk} \; \hat{\beta}^j \hat{\beta}_l \; \epsilon^{lkn} \right) \right] B^n \\
&= \left[\delta^i{}_n + [\gamma - 1] [\delta^i{}_n - \hat{\beta}^i \hat{\beta}_n] \right] B^n \\
&= \left[\gamma [\delta^i{}_n - \hat{\beta}^i \hat{\beta}_n] + \hat{\beta}^i \hat{\beta}_n \right] B^n
\end{aligned}$$

So finally

$$B^i \to \bar{B}^i = \hat{\beta}^i \hat{\beta}_j B^j + \gamma \left\{ [\delta^i{}_j - \hat{\beta}^i \hat{\beta}_j] B^j - (\vec{\beta} \times \vec{E})^i \right\}$$

Note that this is very similar to what happens for the electric field, up to a strategically placed minus sign. In particular, the component of the magnetic field in the direction of motion is not affected by a Lorentz transformation.

Comment: This is actually a rather general effect, it is *extremely* common for a SR formula for magnetic fields to be almost identical to that for electric fields up to strategic minus signs. (You can think of this as a generalization of Lenz' law.) $\diamondsuit$

Collecting the two results we have for a Lorentz transformation along a general direction:

$$E^i \to \bar{E}^i = \hat{\beta}^i \hat{\beta}_j \; E^j + \gamma \; \left\{ (\delta^i{}_j - \hat{\beta}^i \hat{\beta}_j) \; E^j + (\vec{\beta} \times \vec{B})^i \right\}$$

$$B^i \to \bar{B}^i = \hat{\beta}^i \hat{\beta}_j \; B^j + \gamma \; \left\{ (\delta^i{}_j - \hat{\beta}^i \hat{\beta}_j) B^j - (\vec{\beta} \times \vec{E})^i \right\}$$

The final result is not too bad — though the derivation was tedious.

1.12.2 Some exercises:

Exercise: For once I've gone to the trouble of showing you what happens if we Lorentz transform in a general (arbitrary) direction. Now let's simplify life. Show that if we Lorentz transform in the x direction, so that $\beta^i = (\beta, 0, 0)$, that we regain the much simpler looking results

$$\bar{E}^x = E^x; \qquad \bar{E}^y = \gamma(E^y - \beta B^z); \qquad \bar{E}^z = \gamma(E^z + \beta B^y);$$

$$\bar{B}^x = B^x; \qquad \bar{B}^y = \gamma(B^y + \beta E^z); \qquad \bar{B}^z = \gamma(B^z - \beta E^y).$$

Verify that this indeed reproduces the results of our computation based on block multiplication (where we reduced a 4×4 matrix equation to several 2×2 matrix equations). $\diamondsuit$

Exercise: Find the Lorentz transformation properties for the two specific quantities $|\vec{E}|^2 - |\vec{B}|^2$ and $\vec{E} \cdot \vec{B}$. That is evaluate

$$|\vec{E}|^2 - |\vec{B}|^2 \to |\bar{\vec{E}}|^2 - |\bar{\vec{B}}|^2 = ???$$

and

$$\vec{E} \cdot \vec{B} \to \bar{\vec{E}} \cdot \bar{\vec{B}} = ???$$

You should be able to do this easily enough for a Lorentz transformation in an arbitrary direction, but there is no real loss of generality in assuming the Lorentz transformation acts along the x axis. Why? $\diamondsuit$

Exercise: Evaluate the two quantities $F^{ab}\, F_{ab}$ and $\epsilon_{abcd}\, F^{ab}\, F^{cd}$ in terms of $\vec{E}$ and $\vec{B}$. This should make the reason for the results of the previous exercise obvious. Why? $\diamondsuit$

Exercise: The general electromagnetic field is

$$F^{ab} = \frac{1}{c}\left[\begin{array}{c|c} 0 & +\vec{E} \\ \hline -\vec{E} & *B \end{array}\right] = \frac{1}{c}\left[\begin{array}{c|ccc} 0 & +E_x & +E_y & +E_z \\ \hline -E_x & 0 & +B_z & -B_y \\ -E_y & -B_z & 0 & +B_x \\ -E_z & +B_y & -B_x & 0 \end{array}\right]$$

Show that by performing a general Lorentz transformation (boost [change of velocity] plus rotation) you can at any specified point *always* bring it into the standard form

$$F^{ab} = \frac{1}{c}\left[\begin{array}{cc|cc} 0 & +E & 0 & 0 \\ -E & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & +B \\ 0 & 0 & -B & 0 \end{array}\right]$$

and explicitly evaluate the quantities E and B in terms of the quantities $F^{ab}\, F_{ab}$ and $\epsilon_{abcd}\, F^{ab}\, F^{cd}$. $\diamondsuit$

Exercise: Consider a situation where there is no electric field $\vec{E} = 0$ but there is a magnetic field $\vec{B} \neq 0$. A moving electron then experiences a force

$$\vec{F} = q\, \vec{v} \times \vec{B}/c$$

and so does not travel in a straight line, its path is deflected. Now do a Lorentz transformation into the rest frame of the electron. In its rest frame, the 3-velocity of the particle is by definition zero. So in this frame the magnetic force seems to have vanished? What is going on here? Please explain? $\diamondsuit$

Notice that the discussion here has to do with how particles react to an imposed electromagnetic field. I have not yet said anything about how to generate an electromagnetic field.

1.12.3 Lorentz force law

Now let's think about how to get the force law from a variational principle — remember the calculus of variations we talked about in the Mechanix module of Math 321. Pick an action of the form

$$S[X(\lambda)] = -\int m_0 \sqrt{-\eta\left(\frac{\mathrm{d}X}{\mathrm{d}\lambda}, \frac{\mathrm{d}X}{\mathrm{d}\lambda}\right)}\, \mathrm{d}\lambda$$

and apply the Euler–Lagrange equations. Here λ is an arbitrary parameterization of the curve, and because we are in flat space everything is so much simpler than last semester. Also the curve is assumed timelike, so that with our signature conventions $-\eta\,(\mathrm{d}X/\mathrm{d}\lambda,\ \mathrm{d}X/\mathrm{d}\lambda)$ is positive and the square root is guaranteed to be real.

Exercise: Check that the value of $S[X(\lambda)]$ is independent of the choice of parameter. ♢

The Euler–Lagrange equations arising from extremizing this action are

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[m_0 \frac{1}{\sqrt{-\eta\left(\frac{\mathrm{d}X}{\mathrm{d}\lambda}, \frac{\mathrm{d}X}{\mathrm{d}\lambda}\right)}} \frac{\mathrm{d}X}{\mathrm{d}\lambda}\right] = 0.$$

If we choose the parameter λ to be the proper time τ we have the simpler result

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\,(m_0\, V^a) = 0.$$

That is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\, P^a = 0,$$

corresponding to a free particle.

Exercise: What happens if we are dealing with photon, or indeed any particle travelling at the speed of light?
Then there is no elapse of proper time — the invariant interval is always identically zero. Can you construct any sensible version of $S[X(\lambda)]$? ♢

Now let's try something a little more complicated. Consider this action

$$S[X(\lambda)] = \int \left\{ -m_0 \sqrt{-\eta\left(\frac{\mathrm{d}X}{\mathrm{d}\lambda}, \frac{\mathrm{d}X}{\mathrm{d}\lambda}\right)} + q\,\eta\left(A, \frac{\mathrm{d}X}{\mathrm{d}\lambda}\right)\right\} \mathrm{d}\lambda$$

where A is an arbitrary 4-vector, and q is some parameter.

Warning: The $A^a(x)$ occurring here is called the "vector potential". It is not the 4-acceleration of a particle, it is instead a vector field defined on spacetime that we will use

to generate the electromagnetic field F^{ab}. The notation is unfortunately completely standard, and you will have to learn to tell by context whether or not A means 4-acceleration or vector potential.
[And there would be little point to using a for 4-accelerations since then it would be easy to confuse 4-acceleration with 3-acceleration.] $\diamondsuit$

Then the Euler-Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{m_0}{\sqrt{-\eta\left(\frac{\mathrm{d}X}{\mathrm{d}\lambda},\frac{\mathrm{d}X}{\mathrm{d}\lambda}\right)}}\,\frac{\mathrm{d}X}{\mathrm{d}\lambda}+q\;A\right]-q\;\nabla A_\bullet\;\frac{\mathrm{d}X^\bullet}{\mathrm{d}\lambda}=0$$

It is easier to see what is going on if we make the indices explicit:

$$S[X(\lambda)]=\int\left\{-m_0\sqrt{-\eta_{ab}\,\frac{\mathrm{d}X^a}{\mathrm{d}\lambda}\,\frac{\mathrm{d}X^b}{\mathrm{d}\lambda}}+q\;A_a(x)\,\frac{\mathrm{d}X^a}{\mathrm{d}\lambda}\right\}\mathrm{d}\lambda,$$

so that the EOM are:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[m_0\frac{1}{\sqrt{-\eta_{cd}\,\frac{\mathrm{d}X^c}{\mathrm{d}\lambda}\,\frac{\mathrm{d}X^d}{\mathrm{d}\lambda}}}\,\frac{\mathrm{d}X^a}{\mathrm{d}\lambda}+q\;A^a\right]-q\;\nabla^a A_c\,\frac{\mathrm{d}X^c}{\mathrm{d}\lambda}=0$$

Parameterize using proper time τ, this reduces to

$$m_0\frac{\mathrm{d}^2X^a}{\mathrm{d}\tau^2}+q[\nabla_c A^a-\nabla^a A_c]\,\frac{\mathrm{d}X^c}{\mathrm{d}\tau}=0$$

That is

$$m_0\frac{\mathrm{d}V^a}{\mathrm{d}\tau}=q[\nabla^a A^b-\nabla^b A^a]\;V_b$$

So that the action considered above is capable of reproducing the Lorentz force law *but only for those electromagnetic fields that can be written in the form*

$$F^{ab}=\nabla^a A^b-\nabla^b A^a$$

Experimental fact: It is an *observational fact* that all electromagnetic fields we have ever seen in the laboratory are of this type.

$$F^{ab}=\nabla^a A^b-\nabla^b A^a$$

This is equivalent to the statement that we have never yet found a magnetic monopole no matter how hard we have looked. $\diamondsuit$

Exercise: (You may wish to scan through selected portions of the Math 464/Math 465 notes for additional background material. Specifically, look up the chapter on exterior differential forms.)

In the language of differential forms F_{ab} is a two-form which is exact, because it is the exterior derivative of the one-form A_a:

$$F = \mathrm{d}A$$

But because of the Poincare lemma $\mathrm{d}^2 A = 0$, so we know that the electromagnetic fields we observe in the laboratory satisfy

$$\mathrm{d}F = 0$$

This is a 3-form which is Hodge dual to a 1-form, so we know

$$*(\mathrm{d}F) = 0$$

which is a 4-vector equation. If we now decompose the 4-tensor F into $\vec{E}$ and $\vec{B}$, you will obtain a number of partial differential equations constraining $\vec{E}$ and $\vec{B}$. Find these PDEs and interpret them. $\diamondsuit$

Exercise: Show that the statement $\mathrm{d}F = 0$, or equivalently $F = \mathrm{d}A$, is equivalent to the nonexistence of magnetic monopoles.
[This exercise will require you to do a little reading of the physics literature. One suitable way of proceeding is to show that by looking at the space–space components of the 4-tensor equation $F = \mathrm{d}A$ one has $\vec{B} = \vec{\nabla} \times \vec{A}$.] $\diamondsuit$

Exercise: Suppose now that magnetic monopoles do exist so that

$$F^{ab} \neq \nabla^a A^b - \nabla^b A^a$$

Show that in this case we can always find a *pair* of 4-vectors A^a and B^a such that

$$F^{ab} = \nabla^a A^b - \nabla^b A^a + \epsilon^{ab}{}_{cd}[\nabla^c B^d - \nabla^d B^c]$$

This exercise is pure mathematics, and amounts to understanding de Rham's theorem, (which you have *not* seen before), and applying it in a context where the topology is trivial. In 2-form notation

$$F = \mathrm{d}A + *(\mathrm{d}B)$$

You can also view this as a 2-form (3+1)-dimensional generalization of the Helmholtz theorem for a 3-vector field:

$$\vec{b} = \vec{\nabla}\Psi + \vec{\nabla} \times \vec{a}.$$

That is

$$b = \mathrm{d}\Psi + *(\mathrm{d}a).$$

$\diamondsuit$

Exercise: Show that a 1-form (3+1) generalization of the Helmholtz theorem is

$$B = \mathrm{d}\Psi + *(\mathrm{d}\Omega)$$

where B is a 1-form and Ω is a 2-form. Generalize to arbitrary dimensionality and to arbitrary s-forms. This exercise is pure mathematics, with no physics input. $\diamondsuit$

Exercise: For the good of your souls, I will now point out a radically different vector decomposition that is sometimes useful. The Clebsch decomposition writes any arbitrary 3-vector field as

$$\vec{b} = \nabla\phi + \alpha\ \nabla\beta$$

This can also be written as

$$\vec{b} = \nabla\phi + \mathrm{Im}(\psi^*\ \nabla\psi)$$

Prove these results. $\diamondsuit$

Exercise: For a 4-vector the Clebsch decomposition generalizes to

$$B = \mathrm{Im}(\phi^*\ \nabla\phi) + \mathrm{Im}(\psi^*\ \nabla\psi).$$

Prove this. $\diamondsuit$

Research problem: Suppose now that magnetic monopoles exist so that

$$F^{ab} \neq \nabla^a A^b - \nabla^b A^a$$

Find a *clean*, *elegant*, and *useful* variational principle that leads to Euler–Lagrange equations coupling arbitrary particles, ones that can have both electric and magnetic charges, to the general electromagnetic field.

[This is not as easy a problem as it at first seems. There are quite a few *awkward* solutions to this problem in the literature.] $\diamondsuit$

Exercise: Let ρ be the density of electric charge, and $\vec{j}$ the 3-current density. Define a 4-current density

$$J = (\rho, \vec{j})$$

which we can view as a 1-form. Now consider the two equations

$$\mathrm{d}F = 0; \qquad \delta F = J.$$

Show that the single equation $\mathrm{d}F = 0$ is equivalent to the two homogeneous Maxwell equations. Furthermore show that the single equation $\delta F = J$ is equivalent to the two inhomogeneous Maxwell equations.

That is, the Maxwell equations can be written very compactly as

$$\mathrm{d}F = 0; \qquad \delta F = J.$$

Note that this now makes sense in any arbitrary possibly curved spacetime manifold, not just Minkowski space. ♢

Exercise: Consider the action

$$S = \int \left\{ \frac{1}{4} F^{ab} F_{ab} + A_a J^a \right\} \mathrm{d}^4 x$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$. Show that the Euler–Lagrange equation is

$$\frac{\partial}{\partial x^a} F^{ab} = J^b$$

Then re-write the definition of F as $F = \mathrm{d}A$, and deduce $\mathrm{d}F = 0$. Similarly re-write the Euler–Lagrange equation as $\delta F = J$ and so deduce

$$\mathrm{d}F = 0; \qquad \delta F = J.$$

This is now the complete set of Maxwell equations. ♢

Exercise: In terms of the vector potential 1-form A, re-write the complete set of Maxwell equations as

$$\delta\, \mathrm{d}A = J$$

or equivalently

$$* \,\mathrm{d} * \mathrm{d}A = J.$$

♢

Exercise: Show that the Lorenz gauge condition $\dot{A}^0 + \nabla \cdot \vec{A} = 0$, can be written as

$$\delta A = 0$$

and so that in the Lorenz gauge the complete set of Maxwell equations reduce to

$$\Delta A = J$$

where Δ is the Laplace–Beltrami operator

$$\Delta = (\mathrm{d} + \delta)^2 = \mathrm{d}\delta + \delta\mathrm{d}.$$

$\diamondsuit$

Historical note: The Lorenz gauge was apparently first used by the Danish physicist Ludwig Lorenz (1829-1891), though it is commonly misattributed to the Dutch physicist Hendrik Antoon Lorentz (1853-1928). *Many textbooks get the name wrong.*
L. Lorenz, "On the Identity of the Vibrations of Light with Electrical Currents", Philos. Mag. **34** (1867) 287-301.
J. van Bladel, "Lorenz or Lorentz?", IEEE Antennas Prop. Mag. **33** (1991) 69. $\diamondsuit$

1.13 Relativistic scalar potential

A standard tool in non-relativistic mechanics is motion in a potential, where the force is given by

$$\vec{F} = -\nabla\Phi(x).$$

As a basic building block for more general theories, it is useful to develop a relativistic generalization of this situation.

Now for a relativistic scalar potential I could do the most obvious thing and assume the simple force law

$$F^a = -\nabla^a \Phi(x)$$

so that

$$\frac{\mathrm{d}P^a}{\mathrm{d}\tau} = -\nabla^a \Phi(x)$$

But if I choose to do so then the rest mass of the particle is changed by the applied force. Indeed:

$$\frac{\mathrm{d}m_0}{\mathrm{d}t}\, c^2 = -\eta(F,V) = \nabla_a \Phi\, \frac{\mathrm{d}X^a}{\mathrm{d}\tau} = \frac{\mathrm{d}\Phi}{\mathrm{d}\tau}$$

so that

$$m_0\, c^2 - \Phi(x) = \text{constant}$$

Now mathematically, no-one can stop me from postulating such a force law, it's just physically useless. It is simply an observed fact that under all known situations the invariant mass of elementary particles is unchanged by applied forces.[13]

So if I want to introduce a scalar field that respects this observation regarding elementary particles, I will have to force $\eta(F,V) = 0$. One obvious way of doing this is to postulate

$$F^a_{scalar} = -\left[\eta^{ab} + V^a\, V^b\right] \nabla_b \Phi.$$

[13] Though this same criticism cannot be leveled at composite systems.

This now explicitly depends on the 4-velocity of the particle, and the presence of the projection operator $\left[\eta^{ab} + V^a\, V^b\right]$ forces the 4-force to be 4-orthogonal to the 4-velocity.

Exercise: Define

$$h^{ab} = \eta^{ab} + V^a\, V^b$$

or equivalently

$$h^a{}_b = \delta^a{}_b + V^a\, V_b$$

Prove that $h^a{}_b$ really is a projection operator in the sense that

$$h^a{}_b\;\; h^b{}_c = h^a{}_c$$

You will already have seen something extremely similar a few pages back. ♢

Exercise: Consider the action

$$S[X(\lambda)] = \int \left\{ -m_0 \sqrt{-\eta_{ab}\, \frac{\mathrm{d}X^a}{\mathrm{d}\lambda}\, \frac{\mathrm{d}X^b}{\mathrm{d}\lambda}} - \Phi(x) \right\} \mathrm{d}\lambda$$

and find its Euler–Lagrange equations. Parameterize using proper time, and show that this is produces (what we have argued are) the physically inappropriate equations of motion

$$\frac{\mathrm{d}(m_0 V)}{\mathrm{d}\tau} = -\nabla\Phi(x)$$

♢

Exercise: Now consider the *different* action

$$S[X(\lambda)] = \int \left\{ -m_0\, \exp[\Phi(x)/m_0]\, \sqrt{-\eta_{ab}\, \frac{\mathrm{d}X^a}{\mathrm{d}\lambda}\, \frac{\mathrm{d}X^b}{\mathrm{d}\lambda}} \right\} \mathrm{d}\lambda$$

and find its Euler–Lagrange equations. Parameterize using proper time, and show that the Euler–Lagrange equations reduce to

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\exp[\Phi(x)/m_0]\; V^a\right] = -\nabla^a \exp[\Phi(x)/m_0]$$

Then reduce this to (what we have argued are) the physically appropriate equations of motion

$$m_0\, \frac{\mathrm{d}V^a}{\mathrm{d}\tau} = -\left[\eta^{ab} + V^a\, V^b\right] \nabla_b \Phi(x).$$

♢

Exercise: Hence or otherwise deduce that a scalar field with the physically appropriate equations of motion can always be re-interpreted in terms of a conformal distortion of the geometry by defining

$$g_{ab} = \exp[+2\Phi(x)/m_0]\; \eta_{ab}$$

and considering the geodesics of the conformally flat metric g_{ab}.

The "low brow" way of proceeding is to simply plug $g_{ab} = \exp[+2\Phi(x)/m_0]\,\eta_{ab}$ into the formulas for the Christoffel symbols and look at the geodesic equations for g_{ab}. This is however not an efficient way of proceeding, and if you stop and think a little, there is a *much* simpler argument. $\diamondsuit$

Exercise: Find a non-relativistic approximation to

$$S[X(\lambda)] = \int \left\{ -m_0\ \exp[+\Phi(x)/m_0]\ \sqrt{-\eta_{ab}\,\frac{\mathrm{d}X^a}{\mathrm{d}\lambda}\,\frac{\mathrm{d}X^b}{\mathrm{d}\lambda}}\ \right\} \mathrm{d}\lambda$$

Write:

$$\frac{\mathrm{d}X^a}{\mathrm{d}\lambda} = \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}; \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \right)$$

and assume that both $\mathrm{d}x^i/\mathrm{d}\lambda$ and Φ/m_0 are small. To be precise, we take

$$\left| \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \right| \ll \left| \frac{\mathrm{d}t}{\mathrm{d}\lambda} \right|; \qquad \frac{|\Phi|}{m_0} \ll 1.$$

Expand $S[X(\lambda)]$ using the binomial expansion, and show that

$$S[X(\lambda)] = \int \left\{ \frac{1}{2} m_0 \left(\frac{\mathrm{d}x^i}{\mathrm{d}t} \right)^2 - \Phi \right\} \mathrm{d}t + \cdots .$$

Physically interpret this result. $\diamondsuit$

1.14 Relativistic hydrodynamics

In ordinary non-relativistic hydrodynamics one is dealing with two basic equations:

- The equation of continuity
$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho\,\vec{v})$$
relates the density and velocity of the fluid. It is equivalent to the conservation of mass.

- The Euler equation is the fluid mechanics equivalent to Newton's second law, it relates the acceleration of a particle following the flow
$$\vec{a} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v}$$

to the force density $\vec{f}$ and mass density ρ. Specifically

$$\vec{a} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \frac{\vec{f}}{\rho}$$

Relativistic generalizations of these two laws are:

$$\nabla_a(\rho\, V^a) = 0$$

$$A^a = V^b\, \nabla_b V^a$$

where A^a is now a 4-vector *field* of 4-accelerations.

The relativistic continuity equation yields

$$\frac{\partial(\rho\,\gamma)}{\partial t} = \nabla \cdot (\rho\gamma\vec{v})$$

This can be understood by interpreting ρ as proportional to the number density of particles *as measured by an observer moving with the fluid.* That is, ρ is a comoving number density — and it transforms as a scalar. The γ factor then corresponds to the fact that Lorentz contraction "squashes" the fluid in the direction of motion, so that as seen by an observer moving with respect to the fluid the number density of particles is increased to $\rho\,\gamma$.

For the 4-acceleration we have

$$A^i = (\gamma\partial_t + \gamma[\vec{v} \cdot \vec{\nabla}])[\gamma v^i],$$

and

$$A^0 = (\gamma\partial_t + \gamma[\vec{v} \cdot \vec{\nabla}])[\gamma].$$

That is, in terms of the advective derivative:

$$A^i = \gamma \left\{ \partial_t + [\vec{v} \cdot \vec{\nabla}] \right\} [\gamma v^i] = \gamma \frac{\mathrm{d}}{\mathrm{d}t}\, [\gamma v^i],$$

and

$$A^0 = \gamma \left\{ \partial_t + [\vec{v} \cdot \vec{\nabla}] \right\} \gamma = \gamma \frac{\mathrm{d}}{\mathrm{d}t}\, \gamma = \frac{1}{2} \frac{d}{\mathrm{d}t}\, \gamma^2.$$

So at low velocities where $\gamma \to 1$ these clearly reproduce the standard Newtonian results:

$$\vec{A} \to a = \frac{\partial v}{\partial t} + (v \cdot \nabla)v;$$

$$A^0 \to 0.$$

These quantities are also clearly relativistically well defined. For now that's good enough for us. (And if you want more precise justifications, you can always search through [for instance] Landau & Lifshitz, or any other advanced text on fluid mechanics.)

Exercise: Verify that with this definition of the acceleration field $A^a = V^b \nabla_b V^a$ the 4-acceleration is everywhere 4-orthogonal to the 4-velocity. $\diamondsuit$

This statement (and note we are now talking about a fluid, not an isolated particle) holds true independent of the form of the 4-force density and implies that for any relativistic fluid we must *as a matter of principle* enforce

$$\eta(f, V) = 0$$

In particular, for a force arising from a pressure gradient we *cannot* assert

$$\vec{f} = -\nabla p$$

but *must* adopt the relativistic generalization

$$\vec{f^a} = -[\eta^{ab} + V^a\, V^b]\nabla_b p$$

A slightly bizarre feature of relativistic hydrodynamics (not justified at this stage) is that Newton's second law is slightly modified

$$\vec{a} = \frac{\vec{f}}{\rho} \qquad \to \qquad A^a = \frac{f^a}{\rho + p}$$

Effectively the pressure contributes to the inertia. This is enough for now — I'll have considerably more to say about fluids later.

1.15 Summary

As we have seen, the special relativity can be cast into an abstract geometrical form based on the use of a flat 4-manifold called Minkowski space. A lot of the technical machinery of last semester's course on differential geometry can then be used to write special relativity in relatively sophisticated form.

Although much of special relativity can be understood in terms of high school algebra, this is not necessarily the most useful way of doing things — especially when you want to then generalize to the curved spacetime of the general relativity.

Part II

Appendices

Appendix A

Elementary analysis of the special relativistic combination of velocities, Wigner rotation, and Thomas precession

Kane O'Donnell and Matt Visser

Published as:
European Journal of Physics 32 (2011) 1033–1047
doi: 10.1088/0143-0807/32/4/016
arXiv: 1102.2001 [gr-qc]

The purpose of this paper is to provide an elementary introduction to the qualitative and quantitative results of velocity combination in special relativity, including the Wigner rotation and Thomas precession. We utilize only the most familiar tools of special relativity, in arguments presented at three differing levels: (1) utterly elementary, which will suit a first course in relativity; (2) intermediate, to suit a second course; and (3) advanced, to suit higher level students. We then give a summary of useful results, and suggest further reading in this often obscure field.

Keywords: relativistic combination of velocities, Wigner rotation, Thomas precession, special relativity.

A.1 Introduction

The problem of how to consider velocities in a special relativistic setting is fundamental to many areas of both theoretical and applied physics [1, 2, 3, 4, 5, 6, 7]. However, students are rarely introduced to anything beyond the most basic of results (such as the relativistic composition of parallel velocities), on account of the perceived complexity and confusion surrounding the combination of velocities in special relativity. The aim of this paper is to remove some of this confusion, and clarify the qualitative concepts associated with the relativistic combination of velocities, which we do in Section A.2. This includes a description of what such velocities actually "mean", what the Wigner rotation represents, and how this leads to the Thomas precession.

In Section A.3 we provide derivations of certain key quantitative results, using only elementary concepts of special relativity. We begin with the simple cases of relativistically combining parallel and perpendicular velocities in Section A.3.2. This section is particularly relevant for those new to such concepts. The formulae are simple and elementary to derive, yet still illustrate the fundamental issues of combining relativistic velocities, including the Wigner rotation and Thomas precession. In Section A.3.3 we use the results already obtained to consider the general combination of velocities — that is, where the velocities are neither necessarily parallel nor perpendicular. We envisage this section to be suitable for students undertaking a first course in relativity, though proving the results of parts of Section A.3.3 involves extensive vector manipulation.

In Section A.4 we consider the relativistic combination of velocities using the boost matrix formulation of special relativity. Whilst in principle this is no more complex than our elementary derivations of Section A.3, a familiarity with the boost matrix representation is assumed, and hence this section will likely be suitable for students undertaking a second course in relativity.

In Section A.5 we briefly outline how the spinor formulation of special relativity can reproduce the results we have already obtained. This section is only suitable for those already familiar with the spinorial representation of Lorentz transformations, and hence is likely to be accessible mainly for more advanced students.

Lastly, in Section A.6 we give a summary of important (and often equivalent) formulae in this field, and in Section A.7 we provide references for further reading.

A.2 Qualitative introduction

A.2.1 Relativistic combination of velocities

To begin with, consider Alice and Bob, each traveling in a spaceship somewhere in the vicinity of Earth, as illustrated in Figure A.1. Unfortunately, due to equipment malfunction, mission control cannot directly observe the velocity of Bob. Nonetheless, Alice is able to measure the velocity of Bob to be $\vec{v}_2$, and mission control can measure the velocity of Alice to be $\vec{v}_1$. The key question surrounding the relativistic combination of velocities is how we deduce the velocity $\vec{v}_{21}$ of Bob, as seen by mission control, using the velocities $\vec{v}_1$ and $\vec{v}_2$. (Note that from the beginning, we must be clear that $\vec{v}_2$ is measured in Alice's rest frame whilst $\vec{v}_1$ and $\vec{v}_{21}$ are measured in mission control's rest frame.) As shown in Section A.3.3 we may indeed derive a simple formula for this velocity $\vec{v}_{21}$, and it is this quantitative result that embodies what we mean by the relativistic combination of velocities $\vec{v}_1$ and $\vec{v}_2$.

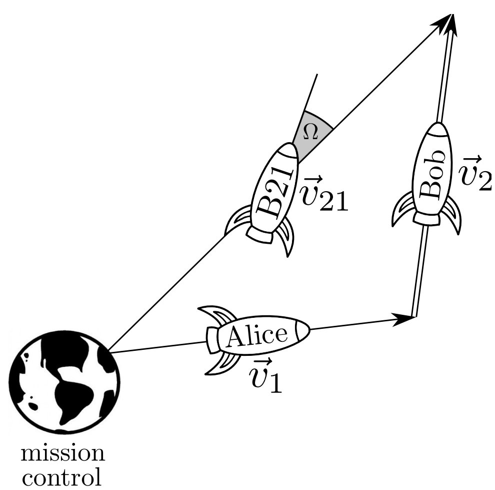

Figure A.1: *The common (and misleading) depiction of the combination of velocities. Mission control sees Alice moving with velocity $\vec{v}_1$, and Alice sees Bob moving with velocity $\vec{v}_2$ (shown as a double line to indicate this is in Alice's frame). Mission control observes Bob as the spacecraft labeled B21, and to be moving at velocity $\vec{v}_{21}$, but pointing in a direction rotated by the Wigner rotation angle Ω. From mission control's perspective, Bob appears to be "sliding" sideways in the direction $\vec{v}_{21}$.*

Note that Figure A.1 is somewhat misleading in that it treats the velocity vectors like Euclidean displacement vectors — in reality, they need not be linked "head-to-tail". Nonetheless this presentation makes it clear which velocities are being combined, and hence remains qualitatively useful.

Wigner rotation

The relativistically combined velocity $\vec{v}_{21}$ cannot be interpreted as directly as we are used to. Whilst mission control observes Bob to be traveling with velocity $\vec{v}_{21}$, they will observe him to be pointing at an angle Ω to $\vec{v}_{21}$, as illustrated in Figure A.1. That is, Bob's frame of reference appears to mission control to be rotated by an angle Ω, which is known as the Wigner rotation angle.

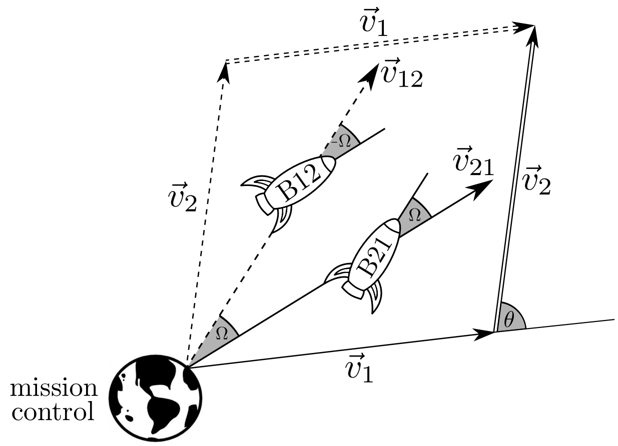

Figure A.2: *A more correct interpretation of the relativistic combination of velocities. The solid lines indicate the case where Alice has velocity $\vec{v}_1$ as measured by mission control, and Bob has velocity $\vec{v}_2$ as measured by Alice, resulting in mission control seeing Bob moving with velocity $\vec{v}_{21}$ (that is, in the spacecraft B21). The dashed lines indicate the naively "symmetrical" case, where Alice has velocity $\vec{v}_2$ as measured by mission control, and Bob has velocity $\vec{v}_1$ as measured by Alice, resulting in mission control seeing Bob moving with velocity $\vec{v}_{12}$ (that is, in the spacecraft B12). In addition, the Wigner rotation angle, $\pm\Omega$ in each case, is also the angle between $\vec{v}_{12}$ and $\vec{v}_{21}$. The angle θ is that between $-\vec{v}_1$ and $\vec{v}_2$, as measured by Alice.*

One may also consider the apparently "symmetrical" case of the combined velocity $\vec{v}_{12}$, where we imagine instead that Bob has a velocity $\vec{v}_1$ as seen by Alice, and Alice has a velocity $\vec{v}_2$ as seen by mission control, as illustrated in Figure A.2. In standard Galilean relativity, we would predict quite rightly that $\vec{v}_{12} = \vec{v}_{21}$. However this is *not* the case when considering the relativistic combination of velocities, as although $||\vec{v}_{12}|| = ||\vec{v}_{21}||$, they do not point in the same direction — there is some angle Ω between them. As we will show, this is also the aforementioned Wigner rotation angle. Indeed, the observation that the Wigner rotation angle corresponds to the angle between $\vec{v}_{12}$ and $\vec{v}_{21}$ can be further extended: In the case of mission control seeing Alice moving with velocity $\vec{v}_1$, and Alice seeing Bob moving with velocity $\vec{v}_2$, then mission control will see Bob traveling at velocity $\vec{v}_{21}$, but pointing in the direction of $\vec{v}_{12}$, as is illustrated in Figure A.2. A similar argument applies for $\vec{v}_{12}$.

Thomas precession

The Thomas precession is a consequence of the Wigner rotation, and arises when one considers the case of Bob experiencing some form of centripetal acceleration. To set up a suitable scenario, let us assume that Alice and Bob are traveling off together to explore the moon, and hence, at some time t, are both traveling at velocity $\vec{v}_1$ as seen by mission control. Hence Alice observes Bob at rest in her frame. This is illustrated in Figure A.3a).

However Bob's boosters suddenly fail, and hence he falls into a circular orbit around Earth, while Alice continues to travel in a straight line toward the moon.[1] Hence Alice now measures Bob to have some velocity $\mathrm{d}\vec{v}_2$ at a later time $\mathrm{d}t$. This is analogous to the situation depicted in Figure A.1, except now $\vec{v}_2$ becomes the infinitesimal $\mathrm{d}\vec{v}_2$. At the time $t + \mathrm{d}t$, we think of Bob's velocity relative to mission control as $\vec{v}_{21} = \mathrm{d}\vec{v}_2 \oplus \vec{v}_1$, and his frame to be rotated by the infinitesimal Wigner rotation angle $\mathrm{d}\Omega$, as illustrated in Figure A.3b). The associated rate of change of the Wigner rotation angle $\mathrm{d}\Omega/\mathrm{d}t$ (that is, how fast Bob's frame is rotating relative to mission control's) is called the Thomas precession rate. The actual Thomas precession Ω_T is the total Wigner rotation turned through if Bob carries out a complete orbit. That is,

$$\Omega_T = \oint_C \frac{\mathrm{d}\Omega}{\mathrm{d}t}\mathrm{d}t \tag{A.1}$$

for any closed curve C in velocity-space.

A.3 Elementary level discussion

A.3.1 Parallel velocities

To begin the discussion, we consider the relativistic combination of velocities for the special cases of parallel and perpendicular velocities $\vec{v}_1$ and $\vec{v}_2$, as illustrated in Fig A.4a) and Fig A.4b) respectively. However, as the relativistic combination of parallel velocities formula is usually given in textbooks (see for example [1] or [2]), we merely state the well-known result:

$$\vec{v}_{21} = \vec{v}_{12} = \frac{\vec{v}_1 + \vec{v}_2}{1 + \vec{v}_1 \cdot \vec{v}_2}\ , \tag{A.2}$$

where we have set $c = 1$. It is important to note that in this case the direction and

[1] Alice is using her boosters so as not to fall into orbit, and hence is not accelerating with respect to mission control. We should also be clear that we are considering the case of Newtonian gravity, and only for the pedagogical purpose of providing a centripetal acceleration in this example. In fact any force applied through the centre of mass would do the job, such as (for example) a string attached to a gimbal at the centre of mass. For an electron one might like to consider an electromagnetic force.

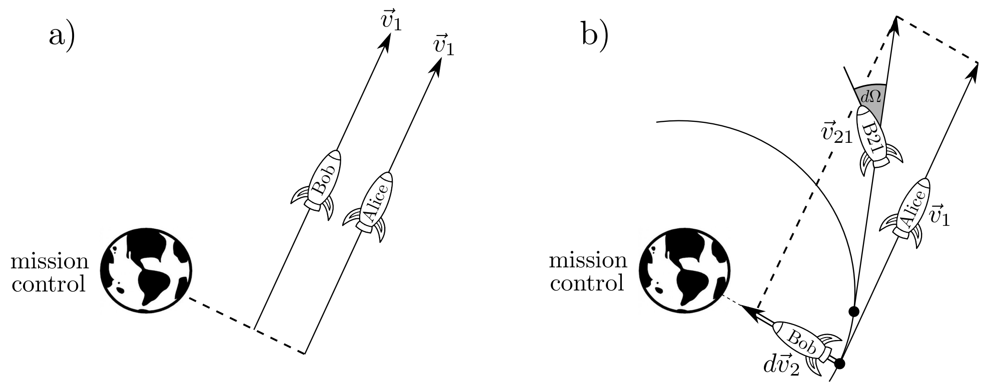

Figure A.3: *At time t mission control sees both Alice and Bob traveling at velocity $\vec{v}_1$, as shown in subfigure a). However Bob's boosters fail, and he falls into Earth's orbit, so at a time* $\mathrm{d}t$ *later, Alice measures Bob to have a velocity* $\mathrm{d}\vec{v}_2$*. Mission control now sees Bob (labeled as spacecraft B21) to be moving with velocity $\vec{v}_{21} = \mathrm{d}\vec{v}_2 \oplus \vec{v}_1$, and his direction rotated by the infinitesimal Wigner rotation angle* $\mathrm{d}\Omega$*, as shown in subfigure b).*

magnitude of the two combined velocities $\vec{v}_{21}$ and $\vec{v}_{12}$ are the same, and hence there will be no resulting Wigner rotation or Thomas precession. It also illustrates that Thomas precession can indeed only occur for centripetal motion, where $\vec{v}_1$ and $d\vec{v}_2$ are not collinear.

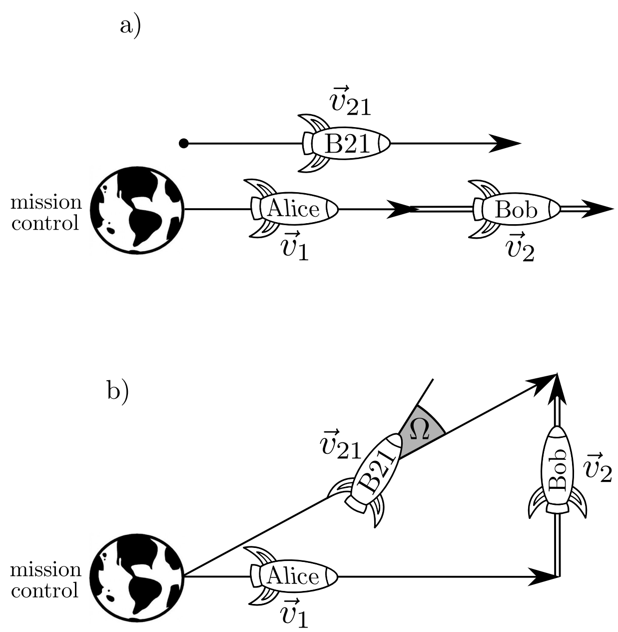

Figure A.4: *Parallel and perpendicular relativistic combination of velocities is illustrated in subfigures a) and b) respectively.*

A.3.2 Perpendicular velocities

The formula for the relativistic combination of perpendicular velocities can be derived in a similar manner as for the parallel case. Here we will use the elementary concepts of time dilation and length contraction. A more explicit Lorentz transformation calculation can easily verify the following results.

Consider the case illustrated in Figure A.4b), where $\vec{v}_1$ and $\vec{v}_2$ are perpendicular.[2] As there is no length contraction for perpendicular distances, but time dilation still occurs in moving from Alice's frame to mission control's, then the velocity $\vec{v}_2$ *in mission control's reference frame* is just $\vec{v}_2/\gamma_1$. Therefore the velocity $\vec{v}_{21}$ of Bob as seen by mission control is just

$$\vec{v}_{21} = \vec{v}_1 + \frac{\vec{v}_2}{\gamma_1} = \vec{v}_1 + \vec{v}_2\sqrt{1 - v_1^2}\ . \tag{A.3}$$

Similarly we find that

$$\vec{v}_{12} = \vec{v}_2 + \frac{\vec{v}_1}{\gamma_2} = \vec{v}_2 + \vec{v}_1\sqrt{1 - v_2^2}\ . \tag{A.4}$$

These formulae are extremely useful to introduce the concept of relativistically combining velocities. They are simple and almost trivial to derive, while still illustrating the fundamental concepts of relativistic velocity combination — the non-intuitive addition laws, the Wigner rotation, and the Thomas precession — as we shall now see.

Wigner rotation

Let us continue with the case of $\vec{v}_1$ and $\vec{v}_2$ being perpendicular, and hence with the relativistic combined velocities $\vec{v}_{21}$ and $\vec{v}_{12}$ as defined by (A.3) and (A.4) respectively. We note that

$$\vec{v}_{21} \neq \vec{v}_{12}, \qquad \text{but} \qquad ||\vec{v}_{12}|| = ||\vec{v}_{21}|| = \sqrt{v_1^2 + v_2^2 - v_1^2 v_2^2}\ . \tag{A.5}$$

As $\vec{v}_{12}$ and $\vec{v}_{21}$ have the same magnitude, but a different direction, we expect some form of Wigner rotation, as previously discussed. One may naively guess that the Wigner rotation angle Ω may have something to do with the angle between $\vec{v}_{21}$ and $\vec{v}_{12}$, and in fact, as we will show in Section A.4.2, it turns out that the angle between $\vec{v}_{21}$ and $\vec{v}_{12}$ is *exactly* the Wigner rotation angle. We can easily calculate this angle by using the definition of the cross product and equations (A.3), (A.4) and (A.5):

$$\sin\Omega = \frac{||\vec{v}_{12} \times \vec{v}_{21}||}{||\vec{v}_{12}||\ ||\vec{v}_{21}||} = \frac{v_1 v_2\left(1 - \frac{1}{\gamma_1\gamma_2}\right)}{v_1^2 + v_2^2 - v_1^2 v_2^2} = \frac{v_1 v_2 \gamma_1\gamma_2}{1 + \gamma_1\gamma_2}\ . \tag{A.6}$$

[2] Note that we must be clear what we mean by "perpendicular," as recall that $\vec{v}_1$ is measured in mission control's rest frame, whilst $\vec{v}_2$ is measured in Alice's rest frame. It only makes sense to say two velocities are perpendicular if they are measured in the same reference frame. Hence when we say that $\vec{v}_1$ and $\vec{v}_2$ are perpendicular, we actually mean that, *in Alice's frame of reference*, the velocity of mission control is $-\vec{v}_1$ and the velocity of Bob is $\vec{v}_2$ and these two velocities are perpendicular.

Again, this is an extremely simple formula for the Wigner rotation angle Ω, which is easily verifiable, using only the fundamental concepts of relativity. While (A.6) only applies in the case of perpendicular velocities, it nonetheless introduces Wigner rotation, and is sufficient for considering the Thomas precession.

Thomas precession

Recall that the Thomas precession rate gives how fast Bob's frame is rotating with respect to mission control's. In our case, Alice sees mission control traveling at $-\vec{v}_1$ and Bob traveling at some infinitesimal velocity $\mathrm{d}\vec{v}_2$, as shown in Figure A.3b). If we assume that Bob is traveling in a circular orbit around Earth, then $\mathrm{d}\vec{v}_2$ is perpendicular to $\vec{v}_1$, and hence our formula for the Wigner rotation angle (A.6) applies. As we let $\mathrm{d}\vec{v}_2 \to 0$, then $\gamma_2 \to 1$ and we find the infinitesimal Wigner rotation angle to first order in $\mathrm{d}v_2$ is (using the small angle approximation)

$$\mathrm{d}\Omega = v_1 \left(\frac{\gamma_1}{1+\gamma_1} \right) \mathrm{d}v_2 \ . \tag{A.7}$$

Hence the Thomas precession rate of Bob's frame as measured by mission control is

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = a v_1 \left(\frac{\gamma_1}{1+\gamma_1} \right) \ , \tag{A.8}$$

where $a = \mathrm{d}v_2/\mathrm{d}t$ is the centripetal acceleration experienced by Bob. Hence we see that, at least for the specific case of circular motion, the formula describing the Thomas precession is simple, with a physically intuitive and elementary derivation.

A.3.3 General velocities

In general, the relativistic combination of velocities in arbitrary directions is nowhere near as simple as in the parallel and perpendicular cases previously discussed. However, we shall now present a derivation of a general formula for $\vec{v}_{21}$ which relies only on the elementary results of (A.2) and (A.3), and a simple time dilation argument. Let us consider the general situation of Figure A.1, however we now decompose Bob's velocity as seen by Alice into its component $\vec{v}_{2\parallel 1}$ parallel to $\vec{v}_1$ and its component $\vec{v}_{2\perp 1}$ perpendicular to $\vec{v}_1$, as illustrated in Figure A.5a). Let S^o denote the rest frame of some contrived intermediate observer, whom Alice measures to have velocity $\vec{v}_{2\parallel 1}$, as in Figure A.5a).

As $\vec{v}_{2\parallel 1}$ and $\vec{v}_1$ are collinear, by (A.2), the velocity of S^o as measured by mission control is

$$\vec{v}^{\,o}_1 = \frac{\vec{v}_{2\parallel 1} + \vec{v}_1}{1 + \vec{v}_{2\parallel 1} \cdot \vec{v}_1} = \frac{\vec{v}_{2\parallel 1} + \vec{v}_1}{1 + \vec{v}_1 \cdot \vec{v}_2} \ , \tag{A.9}$$

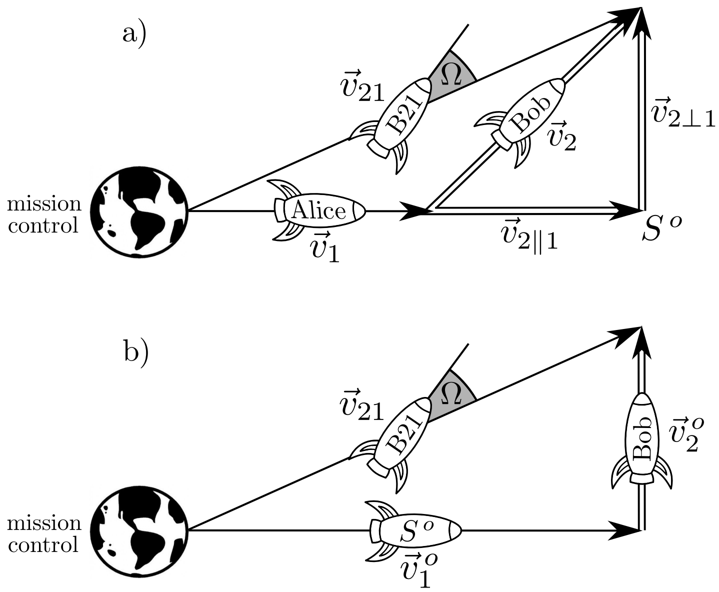

Figure A.5: *In subfigure a) we decompose the velocity $\vec{v}_2$ of Bob as measured by Alice into the components $\vec{v}_{2\parallel 1}$ and $\vec{v}_{2\perp 1}$, parallel and perpendicular to $\vec{v}_1$ respectively. The relativistically combined velocity $\vec{v}_{21}$ is the velocity of Bob as seen by mission control. In subfigure b), we see the S^o frame, which is observed to have velocity $\vec{v}_1^o$ by mission control, which represents the relativistic combination of velocities $\vec{v}_{2\parallel 1}$ and $\vec{v}_1$. In the S^o frame Bob is measured to have velocity $\vec{v}_2^o$, perpendicular to $\vec{v}_1^o$.*

and there is no Wigner rotation of the S^o frame relative to mission control. Therefore, we can think of a new situation, as illustrated in Figure A.5b), where we have S^o moving at velocity $\vec{v}_1^o$ relative to mission control, and Bob moving at some velocity $\vec{v}_2^o$ as measured in the S^o frame. Using arguments similar to those in Section A.3, since $\vec{v}_1^o$ and $\vec{v}_2^o$ are perpendicular, then, due to time dilation

$$\vec{v}_2^o = \gamma_{2\parallel 1}\,\vec{v}_{2\perp 1} \qquad \text{where} \qquad \gamma_{2\parallel 1} = \frac{1}{\sqrt{1 - v_{2\parallel 1}^2}}\,. \tag{A.10}$$

Thus, as mission control sees S^o moving at velocity $\vec{v}_1^o$, and the observer S^o sees Bob to be moving at the perpendicular velocity $\vec{v}_2^o = \gamma_{2\parallel 1}\,\vec{v}_{2\perp 1}$, we may apply formula (A.3) for the relativistic combination of perpendicular velocities. Replacing $\vec{v}_1$ with $\vec{v}_1^o$ and $\vec{v}_2$ with $\vec{v}_2^o = \gamma_{2\parallel 1}\,\vec{v}_{2\perp 1}$ we see that the velocity of Bob with respect to mission control is given by

$$\vec{v}_{21} = \vec{v}_1^o + \frac{\gamma_{2\parallel 1}}{\gamma_1^o}\,\vec{v}_{2\perp 1}\,. \tag{A.11}$$

Furthermore, from (A.9) we see that

$$\gamma_1^o \equiv \frac{1}{\sqrt{1 - (v_1^o)^2}} = \gamma_{2\parallel 1}\gamma_1(1 + \vec{v}_1 \cdot \vec{v}_2)\,, \tag{A.12}$$

and hence (A.11) becomes

$$\vec{v}_{21} \;=\; \frac{\vec{v}_1 + \vec{v}_{2\parallel 1} + \sqrt{1-v_1^2}\;\vec{v}_{2\perp 1}}{1+\vec{v}_1\cdot\vec{v}_2} \;=\; \frac{\vec{v}_2 + \gamma_1\vec{v}_1 + (\gamma_1-1)(\vec{v}_1\cdot\vec{v}_2)\vec{v}_1/v_1^2}{\gamma_1(1+\vec{v}_1\cdot\vec{v}_2)}\,. \tag{A.13}$$

Similarly, we find

$$\vec{v}_{12} \;=\; \frac{\vec{v}_2 + \vec{v}_{1\parallel 2} + \sqrt{1-v_2^2}\;\vec{v}_{1\perp 2}}{1+\vec{v}_1\cdot\vec{v}_2} \;=\; \frac{\vec{v}_1 + \gamma_2\vec{v}_2 + (\gamma_2-1)(\vec{v}_1\cdot\vec{v}_2)\vec{v}_2/v_2^2}{\gamma_2(1+\vec{v}_1\cdot\vec{v}_2)}\,. \tag{A.14}$$

These are the most elementary formulae for the composition of general velocities that we have been able to uncover. Their derivation is simple and fundamental, with an easily attributable physical motivation.

Wigner rotation

Whilst this subsection introduces no new concepts, the vector algebra becomes more tedious and may somewhat confuse the issue, so we consider this subsection to be more suitable for advanced students in a first course on relativity. We use a similar procedure as in Section A.3.2 to consider the Wigner rotation; to do so we must have $||\vec{v}_{21}|| = ||\vec{v}_{12}||$. We leave it to the reader to verify that indeed

$$||\vec{v}_{21}|| = ||\vec{v}_{12}|| = \frac{\sqrt{||\vec{v}_1+\vec{v}_2||^2 - ||\vec{v}_1\times\vec{v}_2||^2}}{1+\vec{v}_1\cdot\vec{v}_2}\,, \tag{A.15}$$

and that this agrees with the parallel and perpendicular cases already discussed. Thus $\vec{v}_{21}$ and $\vec{v}_{12}$ have the same magnitude — but by (A.13) and (A.14) they are not equal, and hence must point in different directions. As previously described for the perpendicular case in Section A.3.2, the Wigner rotation angle Ω is exactly the angle between $\vec{v}_{21}$ and $\vec{v}_{12}$ as measured by mission control. (We shall explicitly prove this in Section A.4.2). To calculate Ω, firstly rewrite (A.13) and (A.14) as

$$\vec{v}_{21} = \frac{\vec{v}_1 + (1-\gamma_1^{-1})\vec{v}_{2\parallel 1} + \gamma_1^{-1}\vec{v}_2}{1+\vec{v}_1\cdot\vec{v}_2} \qquad \text{and} \qquad \vec{v}_{12} = \frac{\vec{v}_2 + (1-\gamma_2^{-1})\vec{v}_{1\parallel 2} + \gamma_2^{-1}\vec{v}_1}{1+\vec{v}_1\cdot\vec{v}_2}\,. \tag{A.16}$$

The Wigner rotation angle Ω then follows from the cross–product of the vectors $\vec{v}_{21}$ and $\vec{v}_{12}$. Using (A.15) and (A.16), this results in

$$\sin(\Omega) = \frac{||\left(\vec{v}_2 + (1-\gamma_2^{-1})\vec{v}_{1\parallel 2} + \gamma_2^{-1}\vec{v}_1\right)\times\left(\vec{v}_1 + (1-\gamma_1^{-1})\vec{v}_{2\parallel 1} + \gamma_1^{-1}\vec{v}_2\right)||}{||\vec{v}_1+\vec{v}_2||^2 - ||\vec{v}_1\times\vec{v}_2||^2}\,, \tag{A.17}$$

which can be simplified to

$$\sin(\Omega) = v_1 v_2\sin\theta\;\; \frac{\left[1-\gamma_1^{-1}\gamma_2^{-1} + (\vec{v}_1\cdot\vec{v}_2)\left(\frac{1}{1+\gamma_1^{-1}}+\frac{1}{1+\gamma_2^{-1}}\right) + \frac{(\vec{v}_1\cdot\vec{v}_2)^2}{(1+\gamma_1^{-1})(1+\gamma_2^{-1})}\right]}{||\vec{v}_1+\vec{v}_2||^2 - ||\vec{v}_1\times\vec{v}_2||^2}\,, \tag{A.18}$$

where θ is the angle between $\vec{v}_1$ and $\vec{v}_2$ as measured by Alice.[3] However

$$\gamma_{12} \equiv \frac{1}{\sqrt{1 - v_{12}^2}} = \gamma_1\gamma_2(1 + \vec{v}_1 \cdot \vec{v}_2) \tag{A.19}$$

may be rearranged to give

$$\cos\theta = \frac{\gamma_{12} - \gamma_1\gamma_2}{v_1 v_2 \gamma_1 \gamma_2} \, . \tag{A.20}$$

Hence, after a little massaging, (A.18) may then be simplified to

$$\sin\Omega = \frac{v_1 v_2 \gamma_1 \gamma_2 (1 + \gamma_1 + \gamma_2 + \gamma_{12})}{(\gamma_1 + 1)(\gamma_2 + 1)(\gamma_{12} + 1)} \sin\theta \, , \tag{A.21}$$

which some may recognize as Stapp's elegant formula [3]. Similarly, using the definition of the dot–product to find the Wigner rotation angle Ω, one finds

$$\cos\Omega = \frac{||\vec{v}_1 + \vec{v}_2||^2 - ||\vec{v}_1 \times \vec{v}_2||^2 \left[\frac{1}{1+\gamma_1^{-1}} + \frac{1}{1+\gamma_2^{-1}} - \frac{\vec{v}_1\cdot\vec{v}_2}{\left(1+\gamma_1^{-1}\right)\left(1+\gamma_2^{-1}\right)}\right]}{||\vec{v}_1 + \vec{v}_2||^2 - ||\vec{v}_1 \times \vec{v}_2||^2} \, , \tag{A.22}$$

and eventually

$$\cos\Omega + 1 = \frac{(\gamma_{12} + \gamma_1 + \gamma_2 + 1)^2}{(\gamma_1 + 1)(\gamma_2 + 1)(\gamma_{12} + 1)} \, . \tag{A.23}$$

Indeed there are many explicit formulae for the Wigner rotation angle Ω, a few of which are given in Section A.6. Stapp's formula arguably remains the simplest and most useful. Note that while the derivation has been somewhat tedious in terms of algebra, the underlying physics is utterly elementary — boiling down to the use of time dilation arguments combined with the usual composition of parallel velocities.

Thomas precession

We consider the same argument as given in Section A.3.2, however we now do not make the simplifying assumption that $\vec{v}_1$ is perpendicular to $\mathrm{d}\vec{v}_2$ — that is, Bob need not be in a circular orbit. It still remains true that the infinitesimal Wigner rotation in mission control's frame of reference is given by letting $\mathrm{d}\vec{v}_2 \to 0$, however we now use our general formula (A.21). Doing so, then $\gamma_2 \to 1$, and from (A.19), we see that $\gamma_{12} \to \gamma_1$. Hence the infinitesimal Wigner rotation angle $\mathrm{d}\Omega$ is, to first degree in $\mathrm{d}\vec{v}_2$,

$$\mathrm{d}\Omega \approx \sin(\mathrm{d}\Omega) = v_1 \mathrm{d}v_2 \sin\theta \, \frac{\gamma_1}{1+\gamma_1} = ||\vec{v}_1 \times \mathrm{d}\vec{v}_2|| \, \frac{\gamma_1}{1+\gamma_1} \, . \tag{A.24}$$

[3] Note that $\vec{v}_1$ and $\vec{v}_2$ are in different frames, so it makes no sense to compare the angle between these velocities. What we really mean is that θ is the angle between $\vec{v}_2$ and the velocity, $-\vec{v}_1$, of mission control, as seen by Alice.

Therefore the Thomas precession rate in mission control's frame of reference is[4]

$$\frac{\mathrm{d}\vec{\Omega}}{\mathrm{d}t} = \vec{v}_1 \times \vec{a}\left(\frac{\gamma_1}{1+\gamma_1}\right) \ , \tag{A.26}$$

where $\vec{a} = \mathrm{d}\vec{v}_2/\mathrm{d}t$ is the centripetal acceleration experienced by Bob. At this stage, it is clear that (A.26) simplifies to the formula (A.8) we obtained in the perpendicular case.

A.4 Intermediate level — boost matrix formulation

We now consider the relativistic combination of velocities using the boost matrix formulation of special relativity. The results derived confirm those already found in Section A.3, however the use of boost matrices gives further conceptual insight — notably that the Wigner rotation angle is the angle between $\vec{v}_{21}$ and $\vec{v}_{12}$.

A.4.1 Composition of boosts

Firstly, consider an arbitrary boost from a frame S to another frame S^o that is moving at a velocity $\vec{v}$ relative to S. Setting $c = 1$, the boost matrix B representing the transformation from the S frame to the S^o frame, such that $\vec{x}^o = B\vec{x}$, is

$$B = \left[\begin{array}{c|c} \gamma & -\gamma\vec{v}^T \\ \hline -\gamma\vec{v} & \mathbb{I} + (\gamma-1)\frac{v_i v_j}{v^2} \end{array}\right] = \left[\begin{array}{c|c} \gamma & -\gamma\vec{v}^T \\ \hline -\gamma\vec{v} & P_v + \gamma Q_v \end{array}\right] \ , \tag{A.27}$$

where we have used the notation P_v to represent the projection onto the plane perpendicular to $\vec{v}$ (explicitly, $[P_v]_{ij} = \delta_{ij} - v_i v_j/v^2$). Similarly $Q_v = \mathbb{I} - P_v$ gives the part parallel to $\vec{v}$. Now, any Lorentz transformation L may be decomposed into a boost followed by a rotation:[5]

$$L = RB \tag{A.28}$$

for some rotation R and some boost B. Furthermore, rotations take the form

$$R = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & R_3 \end{array}\right] \ , \tag{A.29}$$

[4]There is a degree of confusion surrounding the precise definition of the Thomas precession. Our result agrees with that of [6] and [7], and not with the more well known one of [1], which gives the Thomas precession in mission control's frame as

$$\frac{\mathrm{d}\vec{\Omega}}{\mathrm{d}t} = \vec{v}_1 \times \vec{a}\left(\frac{\gamma_1^2}{1+\gamma_1}\right) = \vec{v}_1 \times \vec{a}\left(\frac{\gamma_1 - 1}{v_1^2}\right) \ . \tag{A.25}$$

This however, as explained in [6] and [7], is actually the Thomas precession rate as viewed from Alice's reference frame. The additional γ_1 factor is due to the time dilation between frames.

[5]Or a rotation followed by a boost — we will use the form of (A.28) consistently throughout the paper.

where R_3 is some three-dimensional rotation matrix. Hence by (A.27), (A.28), and (A.29), any Lorentz transformation can be written in the form

$$L = \left[\begin{array}{c|c} \gamma & -\gamma \vec{v}^T \\ \hline -\gamma R_3 \vec{v} & R_3\{P_v + \gamma\, Q_v\} \end{array}\right] . \tag{A.30}$$

Thus we can calculate what the net Lorentz transformation L_{21} is for the situation depicted in Figure A.1 by simply composing the two associated boosts — that is, boosting first by B_1 and then B_2 — so as to move from the S frame to the S^o frame. Hence

$$L_{21} = B_2 B_1 . \tag{A.31}$$

Writing this out explicitly using (A.27), we see that

$$L_{21} = \left[\begin{array}{c|c} \gamma_2 & -\gamma_2 \vec{v}_2^T \\ \hline -\gamma_2 \vec{v}_2 & P_2 + \gamma_2 Q_2 \end{array}\right] \left[\begin{array}{c|c} \gamma_1 & -\gamma_1 \vec{v}_1^T \\ \hline -\gamma_1 \vec{v}_1 & P_1 + \gamma_1 Q_1 \end{array}\right] \tag{A.32}$$

$$= \left[\begin{array}{c|c} \gamma_2\gamma_1(1 + \vec{v}_2 \cdot \vec{v}_1) & -\gamma_2\gamma_1 \vec{v}_1^T - \gamma_2 \vec{v}_2^T [P_1 + \gamma_1 Q_1] \\ \hline -\gamma_2\gamma_1 \vec{v}_2 - \gamma_1 [P_2 + \gamma_2 Q_2] \vec{v}_1 & [P_2 + \gamma_2 Q_2][P_1 + \gamma_1 Q_1] + \gamma_1\gamma_2 \vec{v}_2 \vec{v}_1^T \end{array}\right] . \tag{A.33}$$

Thus if we wish to decompose this Lorentz transformation into the form $L_{21} = RB_{21}$, and we let $\vec{v}_{21}$ denote the velocity corresponding to the boost B_{21}, we can equate (A.30) and (A.33), which gives

$$\left[\begin{array}{c|c} \gamma_{21} & -\gamma_{21} \vec{v}_{21}^T \\ \hline -\gamma_{21} R_3 \vec{v}_{21} & R_3[P_{21} + \gamma\, Q_{21}] \end{array}\right]$$

$$= \left[\begin{array}{c|c} \gamma_2\gamma_1(1 + \vec{v}_2 \cdot \vec{v}_1) & -\gamma_2\gamma_1 \vec{v}_1^T - \gamma_2 \vec{v}_2^T [P_1 + \gamma_1 Q_1] \\ \hline -\gamma_2\gamma_1 \vec{v}_2 - \gamma_1 [P_2 + \gamma_2 Q_2] \vec{v}_1 & [P_2 + \gamma_2 Q_2][P_1 + \gamma_1 Q_1] + \gamma_1\gamma_2 \vec{v}_2 \vec{v}_1^T \end{array}\right] . \tag{A.34}$$

Comparing the 00 terms, we see that

$$\gamma_{21} = \gamma_2\gamma_1(1 + \vec{v}_2 \cdot \vec{v}_1) , \tag{A.35}$$

as we found previously in (A.19) (note that $\gamma_{21} = \gamma_{12}$). Using this result in comparing the $0j$ terms of (A.34), we see that

$$\vec{v}_{21} = \frac{\vec{v}_1 + \gamma_1^{-1} P_1 \vec{v}_2 + Q_1 \vec{v}_2}{1 + \vec{v}_2 \cdot \vec{v}_1} . \tag{A.36}$$

This can be written alternatively as

$$\vec{v}_{21} = \frac{\vec{v}_1 + \vec{v}_{2\|1} + \sqrt{1 - v_1^2}\; \vec{v}_{2\perp 1}}{1 + \vec{v}_2 \cdot \vec{v}_1} , \tag{A.37}$$

which is what was derived in Section A.3.3. Furthermore, as follows from (A.28) and (A.31), we can define a pure rotation matrix R and a pure boost matrix B_{21} such that

$$B_2 B_1 = R B_{21} \ . \tag{A.38}$$

However for the same rotation matrix R, we have

$$B_1 B_2 = (B_2 B_1)^T = (R B_{21})^T = B_{21}^T R^T = B_{21} R^{-1} = R^{-1}(R B_{21} R^{-1}) \ . \tag{A.39}$$

Since $(R B_{21} R^{-1})^T = R B_{21} R^{-1}$, we see that $R B_{21} R^{-1}$ is a pure boost — so we define $B_{12} \equiv R B_{21} R^{-1}$, such that

$$B_1 B_2 = R^{-1} B_{12} \ . \tag{A.40}$$

The results (A.38) and (A.40) verify the interpretation given in Figure A.2, that whilst mission control may measure Bob to be moving with velocity $\vec{v}_{21}$, his frame of reference will be rotated by Ω, and similarly for $\vec{v}_{12}$ (except the rotation will be by $-\Omega$). Furthermore, from (A.40) it follows that

$$R = B_{12} B_2^{-1} B_1^{-1} \ , \tag{A.41}$$

and hence we have explicitly calculated the rotation matrix R. We can "simplify" this further however, by using (A.40) and noting that

$$B_2 B_1 B_1 B_2 = B_{12}^2 \ . \tag{A.42}$$

Thus by (A.41)

$$R = \sqrt{B_2 B_1 B_1 B_2} B_2^{-1} B_1^{-1} \ . \tag{A.43}$$

We can now use the property that the angle Ω rotated by in the rotation R is related to the trace of the rotation matrix via $\mathrm{tr}(R) = 2(1+\cos(\Omega))$, and hence the Wigner rotation angle Ω is

$$\cos\Omega + 1 = \frac{1}{2}\mathrm{tr}\left(\sqrt{B_2 B_1 B_1 B_2} B_2^{-1} B_1^{-1}\right) \ . \tag{A.44}$$

A.4.2 Connecting the angles

In deriving the formulae in Section A.3.2 and Section A.3.3 for the Wigner rotation angle, we assumed that it was the angle between $\vec{v}_{21}$ and $\vec{v}_{12}$. We can now prove this using our boost matrix formulation.

By (A.28) and (A.29), the Wigner rotation angle Ω is just the angle involved in the rotation R, or equivalently, the three-dimensional rotation R_3. However, consider again (A.14) and (A.35), which show that

$$\gamma_{21} \vec{v}_{12} = \gamma_2 \gamma_1 \vec{v}_2 + \gamma_1 [P_2 + \gamma_2 Q_2] \vec{v}_1 \ . \tag{A.45}$$

By equating the $i0$ entries of (A.34), we see that $\gamma_{21} R_3 \vec{v}_{21}$ is also equal to the right-hand-side of (A.45), and hence

$$R_3 \vec{v}_{21} = \vec{v}_{12} \ . \tag{A.46}$$

Therefore, as previously claimed and now proved, we can find the Wigner rotation angle Ω by simply calculating the angle between $\vec{v}_{12}$ and $\vec{v}_{21}$.

A.4.3 Inverting the transformations

As a final consideration, what is the velocity of mission control as seen by Bob? If mission control sees Bob moving with velocity $\vec{v}_{21}$, does Bob see mission control moving with velocity $-\vec{v}_{21}$ as would be expected in Galilean relativity? If Bob's frame is considered as the observer frame, then he will see Alice moving with velocity $-\vec{v}_2$ and Alice will see mission control moving with velocity $-\vec{v}_1$. Hence the velocity of mission control as observed by Bob is given by the composition of the two velocities $-\vec{v}_2$ and then $-\vec{v}_1$, or in boost matrix form

$$B_{-1}B_{-2} = B_1^{-1}B_2^{-1} = (B_2B_1)^{-1} = (RB_{21})^{-1} = B_{21}^{-1}R^{-1} \ , \tag{A.47}$$

where we have used that $B_{-v} = B_v^{-1}$, i.e. the inverse of a boost in a direction $\vec{v}$ is just a boost in the direction $-\vec{v}$. However from our definition of $B_{12} \equiv RB_{21}R^{-1}$, we see that $B_{21} = R^{-1}B_{12}R$ and hence $B_{21}^{-1} = R^{-1}B_{12}^{-1}R$, so (A.47) implies that

$$B_{-1}B_{-2} = R^{-1}B_{12}^{-1} \ . \tag{A.48}$$

Thus the transformation from Bob's frame to mission control's frame is given by

$$B_{-1}B_{-2} = B_{-21}R^{-1} = R^{-1}B_{-12} \ , \tag{A.49}$$

where B_{-12} and B_{-21} correspond to boosts in the $-\vec{v}_{12}$ and $-\vec{v}_{21}$ directions respectively. However which one, $B_{-21}R^{-1}$ or $R^{-1}B_{-12}$, should we consider to determine the velocity of mission control as observed by Bob? The transformation $B_{-21}R^{-1}$ implies *first* rotating Bob's frame, and *then* boosting along $-\vec{v}_{21}$ to end up in mission control's frame. Hence *in Bob's rotated frame* he will see mission control traveling at velocity $-\vec{v}_{21}$. However the transformation $R^{-1}B_{-12}$ implies first boosting from Bob's frame by $-\vec{v}_{12}$, and *then* rotating. Therefore in Bob's *original frame*, he will see mission control traveling with velocity $-\vec{v}_{12}$ (but pointing in a direction rotated by $-\Omega$, due to Wigner rotation). Thus while mission control sees Bob moving with velocity $\vec{v}_{21}$ in their frame of reference, Bob sees mission control moving with velocity $-\vec{v}_{12}$ in his.

A.5 Advanced level — spinor formulation

In this section we use the spinor formulation of special relativity to consider the relativistic combination of velocities and the Wigner rotation angle. There are in fact two methods of finding the results we require. The first is straightforward but tedious, so we only present the initial formulation, and the final results. The second version is less apparent, but much quicker, and we give the full derivation.

A.5.1 Explicit approach

Let $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ be a 3-vector of Pauli sigma matrices, and let

$$\mathbf{X} = ctI + \mathbf{x}\cdot\boldsymbol{\sigma} \tag{A.50}$$

be a representation of a 4-vector $\mathbf{X} = (ct, \mathbf{x})$ in terms of a Hermitian 2×2 matrix. Then boosts are represented by

$$\mathbf{X} \to \mathbf{BXB}; \qquad \mathbf{B} = \cosh(\xi/2) + \sinh(\xi/2)\ \mathbf{n}\cdot\boldsymbol{\sigma}\ ; \tag{A.51}$$

and rotations by

$$\mathbf{X} \to \mathbf{RXR}^{-1}; \qquad \mathbf{R} = \cos(\theta/2) + i\sinh(\theta/2)\ \mathbf{n}\cdot\boldsymbol{\sigma}\ . \tag{A.52}$$

Where ξ is the rapidity parameter defined by $v = \tanh\xi$. The Wigner rotation is now encoded in the fact that

$$\mathbf{B}_2\mathbf{B}_1 = \mathbf{R}_{21}\mathbf{B}_{21}\ . \tag{A.53}$$

One can write this out explicitly. Defining $\vec{\Omega} = \Omega\,\vec{n}_\Omega$, upon equating coefficients, we find the following four simultaneous, independent equations:

$$\mathbf{n}_\Omega\cdot\mathbf{n}_{12} = 0\ , \tag{A.54}$$

$$\cos\frac{\Omega}{2}\cosh\frac{\xi_{12}}{2} = \cosh\frac{\xi_2}{2}\cosh\frac{\xi_1}{2} + \sinh\frac{\xi_2}{2}\sinh\frac{\xi_1}{2}\ \mathbf{n}_2\cdot\mathbf{n}_1\ , \tag{A.55}$$

$$\begin{aligned}\cos\frac{\Omega}{2}\sinh\frac{\xi_{12}}{2}\ \mathbf{n}_{12} - \sin\frac{\Theta}{2}\sinh\frac{\xi_{12}}{2}\mathbf{n}_\Omega\times\mathbf{n}_{12}&\\ = \cosh\frac{\xi_2}{2}\sinh\frac{\xi_1}{2}\ \mathbf{n}_1 + \cosh\frac{\xi_1}{2}\sinh\frac{\xi_2}{2}\ \mathbf{n}_2\ ,&\end{aligned} \tag{A.56}$$

$$\cosh\frac{\xi_{12}}{2}\sin\frac{\Omega}{2}\ \mathbf{n}_\Omega = \sinh\frac{\xi_1}{2}\sinh\frac{\xi_2}{2}\ \mathbf{n}_1\times\mathbf{n}_2\ . \tag{A.57}$$

One can then test one's algebraic skill and fortitude, to eventually arrive at the already proven results for the Wigner rotation:

$$\cos\Omega + 1 = \frac{(\gamma_{12}+\gamma_1+\gamma_2+1)^2}{(\gamma_1+1)(\gamma_2+1)(\gamma_{12}+1)}\ , \tag{A.58}$$

and the familiar formula of Stapp [3]

$$\sin\Omega = \frac{v_1\gamma_1 v_2\gamma_2(1+\gamma_1+\gamma_2+\gamma_{12})}{(\gamma_1+1)(\gamma_2+1)(\gamma_{12}+1)}\sin\theta\ . \tag{A.59}$$

We congratulate those who verify this procedure! All the physics is already encoded in the equations above — the only difficulty lies in the tedious nature of the algebra.

A.5.2 A more efficient approach

Whilst this derivation is significantly shorter step-wise, it involves some not entirely obvious leaps of understanding that we leave for the reader to verify. To begin with

$$\mathbf{B}_{12}^2 = \mathbf{B}_1\mathbf{B}_2\mathbf{B}_2\mathbf{B}_1 \ , \tag{A.60}$$

and hence

$$\gamma_{12} = \frac{1}{2}\mathrm{tr}(\mathbf{B}_{12}^2) = \frac{1}{2}\mathrm{tr}(\mathbf{B}_1\mathbf{B}_2\mathbf{B}_2\mathbf{B}_1) = \frac{1}{2}\mathrm{tr}(\mathbf{B}_1^2\mathbf{B}_2^2) = \gamma_1\gamma_2(1+\mathbf{v}_1\cdot\mathbf{v}_2) \ . \tag{A.61}$$

This then leads to

$$\cos\theta = \frac{\gamma_{12}-\gamma_1\gamma_2}{\sqrt{(\gamma_1^2-1)(\gamma_2^2-1)}} \ . \tag{A.62}$$

Using these results, and the fact that $\mathrm{tr}(\mathbf{R}_{12}\mathbf{B}_{12}) = \mathrm{tr}(\mathbf{B}_1\mathbf{B}_2)$, we find

$$\sqrt{1+\cos\Omega}\sqrt{\gamma_{12}+1} = \sqrt{(\gamma_1+1)(\gamma_2+1)} + \sqrt{(\gamma_1-1)(\gamma_2-1)}\cos\theta \ . \tag{A.63}$$

Thus

$$\cos\Omega + 1 = \frac{(1+\gamma_1+\gamma_2+\gamma_{12})^2}{(\gamma_1+1)(\gamma_2+1)(\gamma_{12}+1)} \tag{A.64}$$

as required.

A.6 Summary of useful formulae

General: The relativistic combination of general velocities $\vec{v}_1$ and $\vec{v}_2$:

$$\vec{v}_{21} = \frac{\vec{v}_1 + \vec{v}_{2\|1} + \sqrt{1-v_1^2}\ \vec{v}_{2\perp 1}}{1+\vec{v}_1\cdot\vec{v}_2} = \frac{\vec{v}_2 + \gamma_1\vec{v}_1 + (\gamma_1-1)(\vec{v}_1\cdot\vec{v}_2)\vec{v}_1/v_1^2}{\gamma_1(1+\vec{v}_1\cdot\vec{v}_2)} \ , \tag{A.65}$$

$$\vec{v}_{12} = \frac{\vec{v}_2 + \vec{v}_{1\|2} + \sqrt{1-v_2^2}\ \vec{v}_{1\perp 2}}{1+\vec{v}_1\cdot\vec{v}_2} = \frac{\vec{v}_1 + \gamma_2\vec{v}_2 + (\gamma_2-1)(\vec{v}_1\cdot\vec{v}_2)\vec{v}_2/v_2^2}{\gamma_2(1+\vec{v}_1\cdot\vec{v}_2)} \ , \tag{A.66}$$

$$||\vec{v}_{21}|| = ||\vec{v}_{12}|| = \frac{\sqrt{||\vec{v}_1+\vec{v}_2||^2 - ||\vec{v}_1\times\vec{v}_2||^2}}{1+\vec{v}_1\cdot\vec{v}_2} \ , \tag{A.67}$$

$$\gamma_{12} = \gamma_1\gamma_2(1+\vec{v}_1\cdot\vec{v}_2) \ . \tag{A.68}$$

The Wigner rotation angle Ω:

$$\sin\Omega = \frac{v_1\gamma_1 v_2\gamma_2(1+\gamma_1+\gamma_2+\gamma_{12})}{(\gamma_1+1)(\gamma_2+1)(\gamma_{12}+1)}\sin\theta \ , \tag{A.69}$$

$$\cos\Omega + 1 = \frac{(\gamma_{12}+\gamma_1+\gamma_2+1)^2}{(\gamma_1+1)(\gamma_2+1)(\gamma_{12}+1)} \,. \tag{A.70}$$

The Thomas precession as seen in mission control's reference frame:

$$\frac{\mathrm{d}\vec{\Omega}}{\mathrm{d}t} = \vec{v}_1 \times \vec{a} \left(\frac{\gamma_1}{1+\gamma_1}\right) \,. \tag{A.71}$$

The Thomas precession as seen in Alice's reference frame:

$$\frac{\mathrm{d}\vec{\Omega}}{\mathrm{d}t} = \vec{v}_1 \times \vec{a} \left(\frac{\gamma_1^2}{1+\gamma_1}\right) \,. \tag{A.72}$$

Perpendicular: The relativistic combination of perpendicular velocities $\vec{v}_1$ and $\vec{v}_2$ is particularly elegant:

$$\vec{v}_{21} = \vec{v}_1 + \sqrt{1-v_1^2}\,\vec{v}_2 \,, \tag{A.73}$$

$$\vec{v}_{12} = \vec{v}_2 + \sqrt{1-v_2^2}\,\vec{v}_1 \,, \tag{A.74}$$

$$||\vec{v}_{21}|| = ||\vec{v}_{12}|| = \sqrt{v_1^2+v_2^2-v_1^2v_2^2} \,, \tag{A.75}$$

$$\gamma_{12} = \gamma_1\gamma_2 \,. \tag{A.76}$$

The Wigner rotation angle Ω:

$$\sin\Omega = \frac{v_1\gamma_1v_2\gamma_2}{\gamma_1\gamma_2+1} \,, \tag{A.77}$$

$$\cos\Omega + 1 = \frac{(\gamma_1+1)(\gamma_2+1)}{\gamma_1\gamma_2+1} \,. \tag{A.78}$$

A.7 Further reading

For those students interested in more details regarding the relativistic combination of velocities from a reasonably elementary viewpoint, the explicit boost composition approach taken in reference [5] may prove useful. There are then many other standard textbook approaches, such as can be found in [1] or [2]. Some of the finer details about the relativistic combination of velocities, especially the relationships between the different frames, can be found in references [4] and [7].

However the issue that receives the most attention in the literature is the Thomas precession, (and to a lesser extent the associated Wigner rotation) — partly due to the confusion surrounding it. We feel that readers further interested in the Thomas precession (and relativistic velocity combination in general), will benefit greatly from reference [6], which gives both a review of the literature (where the reader can find many higher-level approaches outlined), a select few of the possible derivations of the Thomas precession formula and some physical interpretations, and it also clarifies some of the misconceptions surrounding the Thomas precession. One such of these is what actually is the correct formulation, as alluded to in Section A.3.3. Reference [7] provides further discussion on this point.

Bibliography

[1] C. Møller. *The Theory of Relativity.* (Oxford University Press, London, 1952). A.1, A.3.1, 4, A.7

[2] J. D. Jackson. *Classical Electrodynamics, 3rd Ed.* (Wiley, New York, 1998). A.1, A.3.1, A.7

[3] H. P. Stapp. "Relativistic theory of polarization phenomena." *Physical Review*, **103** (1956) 425–434. A.1, A.3.3, A.5.1

[4] G. P. Fisher. "Thomas precession." *American Journal of Physics*, **40** (1972) 1772. A.1, A.7

[5] M. Ferraro, R. Thibeault. "Generic composition of boosts: an elementary derivation of the Wigner rotation." *European Journal of Physics*, **20** (1999) 143. A.1, A.7

[6] G. B. Malykin. "Thomas precession: correct and incorrect solutions." *Physics–Uspekhi*, **49** (2006) 837–853. A.1, 4, 4, A.7

[7] V. I. Ritus. "On the difference between Wigner's and Møller's approaches to the description of Thomas precession." *Physics–Uspekhi*, **50** (2007) 95–101. A.1, 4, 4, A.7

Appendix B

Inertial frames without the relativity principle

Valentina Baccetti, Kyle Tate, and Matt Visser

arXiv:1112.1466 [gr-qc] — December 2011

Published as: JHEP **1205** (2012) 119;   doi: 10.1007/JHEP05(2012)119

Abstract: Ever since the work of von Ignatowsky circa 1910 it has been known (if not always widely appreciated) that the relativity principle, combined with the mild physical assumption of linearity and the less mild assumption of isotropy, leads almost uniquely to either the Lorentz transformations of special relativity or to Galileo's transformations of classical Newtonian mechanics. Consequently, if one wishes (for whatever reason) to entertain the possibility of Lorentz symmetry breaking, then it seems likely that one will have to abandon or at the very least grossly modify the relativity principle. We reassess the notion of spacetime transformations between inertial frames in the absence of the relativity principle, arguing that significant nontrivial physics can still be extracted as long as the transformations are at least linear. An interesting technical aspect of the analysis is that the transformations now form a *groupoid/pseudo-group* — it is this technical point that permits one to evade the von Ignatowsky argument. Even in the absence of a relativity principle we can nevertheless deduce clear and compelling rules for the transformation of space and time, rules for the composition of 3-velocities, and rules for the transformation of energy and momentum. The energy-momentum transformations are in general *affine*, but may be *chosen* to be linear, with the 4-component vector $P = (E, -\boldsymbol{p}^T)$ transforming as a row vector, while the 4-component vector of space-time position $X = (t, \boldsymbol{x}^T)^T$ transforms as a column vector. As part of the analysis we identify two particularly elegant and compelling models implementing "minimalist" violations of Lorentz invariance — in one of these minimalist models all Lorentz violations are confined to the neutrino sector, while the second minimalist Lorentz-violating model depends on one free function of absolute velocity, but otherwise preserves as much as possible of standard Lorentz invariant physics. Consequently in many ways these models serve as a "gold standard" when studying possible violations of Lorentz invariance.

Keywords: Inertial frames, relativity principle, kinematics without relativity, Lorentz symmetry breaking.

B.1 Introduction

In 1910 von Ignatowsky established a very tight connection between the group structure implied by the relativity principle and the rules for the transformation of space-time coordinates [1, 2, 3, 4]. Under suitable hypotheses, the relativity principle almost uniquely leads to either the Lorentz transformations or Galileo's transformations, and this result makes no *a priori* appeal to the constancy of the speed of light. Over the last century this same result has been repeatedly rediscovered, expanded upon, and re-analyzed, with significant pedagogical efforts being expended; see for instance [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. The relevance of von Ignatowsky's analysis for our current purposes comes from reversing the logic: If for whatever reason one wishes to speculate about a possible breakdown of Lorentz invariance at ultra-high energies, then one is almost certain to be forced to abandon (or at the very least grossly modify) the relativity principle — and in particular one will in general be forced to abandon the group structure for the set of transformations that connect space and time in one inertial frame to space and time in a different inertial frame.

This is the basic theme of this article: *What happens to inertial frames, and the transformations between inertial frames, if you do not have the relativity principle?* We shall see that quite a lot can still be said. Under very mild hypotheses, it is possible to argue that the transformation rules between inertial frames should at least be linear. A rather general formula for the transformation of 3-velocities, and in particular the composition of 3-velocities can then be derived. We shall see that in theories with a preferred frame (an "aether") the set of transformations between inertial frames forms a *groupoid/pseudogroup*. (In this particular sub-branch of mathematics the mathematical terminology is not 100% settled.) This *groupoid/pseudogroup* structure is not just a mathematical curiosity, the distinction between a *groupoid/pseudogroup* and a *group* is essential to evading von Ignatowsky's argument.

In reference [25] one of the current authors had, with collaborators, explored the possibility of, in certain circumstances, permitting "faster than light" signals while still retaining the relativity principle. In the current article we shall more radically explore what happens if the relativity principle is sacrificed.

Now, "merely" knowing how space and time transform is not sufficient to do anything beyond the most basic of kinematics. A key ingredient to understanding dynamics in Lorentz violating theories is to understand how energy and momentum transform. This is more subtle than one might naively expect. The generic situation is that energy and momentum transform in an *affine* manner (that is, linear plus an inhomogeneous offset term). We show that it is possible, but not always desirable, to choose conventions and parameters in such a way as to force the offset to be zero — in which case energy and momentum transform in a homogeneous linear manner. In fact, if this is done, then with our conventions the 4-component vector $P = (E, -\boldsymbol{p}^T)$, the 4-momentum, is a row vector. P is an element of the vector space dual to the 4-component vector $X = (t, \boldsymbol{x}^T)^T$, the 4-position, which is a column vector. (We shall see that the offset term in the affine transformation is needed if one wishes to recover the usual naive form of Newtonian mechanics in a suitable limit, but that there is a somewhat non-standard formulation of Newtonian mechanics in which energy-momentum can be made to transform linearly.)

Finally, as a side effect of the general analysis, we focus specific attention on two particularly elegant and compelling models implementing a "minimalist" violation of Lorentz invariance. The first minimalist Lorentz-violating model confines all Lorentz violating physics to the neutrino sector. The second minimalist Lorentz-violating model preserves as much as possible of standard Lorentz invariant physics, but the transformations additionally depend on one extra function, an arbitrary free function of absolute velocity. Consequently, when studying possible violations of Lorentz invariance, these two models in many ways serve as a "gold standard" for least-damaged versions of Lorentz invariance.

The considerations of this article will be essential to almost any form of violation of Lorentz invariance that encodes "preferred frame" (aether frame) effects. Certain variants of DSR (doubly special relativity, distorted special relativity) fall outside this framework, others do not [26, 27, 28, 29, 30].

B.2 Why violate Lorentz invariance?

Why is there currently such significant interest in the possibility of broken Lorentz invariance? There are several reasons, based on a variety of considerations. There are purely theoretical speculations, there is the practical need for a phenomenological framework within which to formulate empirical tests of Lorentz invariance, and there are even *tentative* experimental hints of the observation of violations of Lorentz invariance.

Theoretical considerations: There have been numerous and long-standing theoretical suggestions to the effect that quantum gravity might eventually violate Lorentz invariance at ultra-high energies. For instance, there is the string-inspired theoretical framework for characterizing possible violations of Lorentz invariance developed by Kostelecky and collaborators [31, 32, 33, 34, 35, 36, 37, 38, 39]. More recently, the Horava gravity framework [40] naturally includes Lorentz violation [41, 42, 43, 44, 45, 46]. The "analogue spacetime" programme also very naturally leads to models where Lorentz invariance is violated at one level or another [47, 48, 49, 50]. There is also the flat-space non-gravity framework developed by Anselmi [51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63], where Lorentz invariance breaking is used to partially regulate QFT ultraviolet divergences. Further afield, Nielsen and collaborators have studied the renormalization group flow of Lorentz symmetry violating operators in generic QFTs, demonstrating that Lorentz invariance is often an infrared fixed point of a generic Lorentz violating QFT [64, 65, 66, 67, 68]. That is, there is an already vast literature regarding possible violations of Lorentz symmetry, and we have necessarily had to be rather selective in choosing citations.

Phenomenological considerations: If one wishes to observationally test Lorentz invariance one needs some coherent framework to work in that at least allows one to formulate appropriate questions. Over the last decade significant progress along these lines has been made. See for instance work by Coleman and Glashow [69, 70], Jacobson, Liberati, and Mattingly [71, 72, 73, 74, 75, 76, 77, 78, 79, 80], and especially the Living Review by Mattingly [81]. The net result is that we now have a considerable quantity of observational bounds, some of them very stringent observational bounds, constraining the possibility of Lorentz symmetry breaking — although it should perhaps be noted that these analyses are performed in the preferred (aether) frame.

Experimental considerations: The OPERA experiment has recently announced *tentative* but statistically significant evidence for Lorentz symmetry violation in the form of "faster than light" neutrinos [82]. (See also earlier even more tentative results from the MINOS collaboration [83].) In the resulting firestorm, over 160 theoretical articles have been generated in some 11 weeks. Notable contributions include [84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103]. We particularly wish to emphasise the importance of the experimental/observational bounds presented in the Cohen–Glashow [86] and Maccione–Liberati–Mattingly articles [100].

Given this level of interest in the topic, we have feel that it is interesting, useful, and timely to perform a careful analysis of the general and very basic notion of inertial frames in the *absence* of Lorentz invariance. We shall focus particularly on "preferred frame" (aether) versions of Lorentz symmetry breaking — that is, we shall study inertial frames in the *absence* of the relativity principle.

B.3 General transformations between inertial frames

B.3.1 Definition of an inertial frame

Essentially everyone would agree on this characterization of inertial frames:

- All inertial frames are in a state of constant, rectilinear motion with respect to one another; they are not accelerating (in the sense of proper acceleration that would be detected by an accelerometer).
- In an inertial reference frame, the laws of mechanics take their simplest form.
- In an inertial frame, Newton's first law (the law of inertia) is satisfied: Any free motion has a constant magnitude and direction.

If *in addition* you accept the relativity principle, then:

- Physical laws take the same form in all inertial frames.

But, as we have argued above, for some purposes the relativity principle is overkill. And that is the topic we will now explore.

B.3.2 Argument for linearity

We still want the transformations to be linear. By definition an inertial particle, in an inertial frame, is not accelerating

$$\frac{d^2\boldsymbol{x}}{dt^2} = 0; \qquad \boldsymbol{x}(t) = \boldsymbol{x}_0 + \boldsymbol{v}_0 t. \tag{B.1}$$

In any other inertial frame, the particle is again by definition not accelerating

$$\frac{d^2\bar{\boldsymbol{x}}}{d\bar{t}^2} = 0; \qquad \bar{\boldsymbol{x}}(\bar{t}) = \bar{\boldsymbol{x}}_0 + \bar{\boldsymbol{v}}_0 \bar{t}. \tag{B.2}$$

Whatever the transformation is between the two sets of time and space coordinates $\{t, \boldsymbol{x}\}$ and $\{\bar{t}, \bar{\boldsymbol{x}}\}$, the transformation has to map straight lines into straight lines — which forces the transformation to be, at the very worst, projective [23, 24]. By additionally requiring that events in a bounded region map into a bounded region this is actually enough to force the transformations to be linear [23, 24]. That is, writing the 4-position as

$$X = \begin{pmatrix} t \\ \boldsymbol{x} \end{pmatrix}, \tag{B.3}$$

we want

$$X \to \bar{X} = M\, X. \tag{B.4}$$

Note that we adopt conventions where both 3-vectors $\boldsymbol{x}$ and 4-vectors X are column vectors. This minimizes the number of special case fiddles we have to adopt later in the discussion.

Note that the primary physics input is the *observation* that inertial frames exist, and from extremely basic notions of kinematics this is enough to argue for linearity. If one additionally (as we shall see later in the article) wants to develop some notion of Lagrangian/Hamiltonian dynamics, then the observation that free inertial particles exist, coupled with Noether's theorem, can be used to argue for the homogeneity of space and time. Some authors prefer to start from homogeneity, and thereby deduce linearity. There are minor technical issues, but for all practical purposes spacetime homogeneity implies and is implied by linearity of the transformations between inertial frames.

B.3.3 General representation of inertial transformations

Taking linearity as given:

- In the special case of Newtonian physics (Galilean relativity) we have
$$M = \left[\begin{array}{c|c} 1 & \mathbf{0}^T \\ \hline -\boldsymbol{v} & I \end{array}\right]. \tag{B.5}$$
- In the case of Einstein physics (special relativity) we have
$$M = \left[\begin{array}{c|c} \gamma & -\gamma \boldsymbol{v}^T/c^2 \\ \hline -\gamma\boldsymbol{v} & \gamma \boldsymbol{n}\boldsymbol{n}^T + [I - \boldsymbol{n}\boldsymbol{n}^T] \end{array}\right], \tag{B.6}$$
with $\boldsymbol{v} = v\boldsymbol{n}$ and $\gamma = (1 - v^2/c^2)^{-1/2}$. One can also write this as
$$M = \left[\begin{array}{c|c} \gamma & -\gamma \boldsymbol{v}^T/c^2 \\ \hline -\gamma\boldsymbol{v} & \gamma \boldsymbol{n}\otimes\boldsymbol{n} + [I - \boldsymbol{n}\otimes\boldsymbol{n}] \end{array}\right]. \tag{B.7}$$
- Carroll kinematics ("Alice in wonderland kinematics") is a rarely encountered and somewhat unphysical limit of the Lorentz group where one takes $c \to 0$ and $v \to 0$ while keeping the "slowness" $u = v/c^2$ fixed. The resulting transformations
$$M = \left[\begin{array}{c|c} 1 & -\boldsymbol{u}^T \\ \hline \mathbf{0} & I \end{array}\right] \tag{B.8}$$
correspond to [104, 105, 106, 107, 108]:
$$t \to \bar{t} = t - \boldsymbol{u}\cdot\boldsymbol{x}; \qquad \boldsymbol{x} \to \bar{\boldsymbol{x}} = \boldsymbol{x}. \tag{B.9}$$
We will have very little to say concerning this particular option.

In the general case, we only know that M is *some* matrix which we can, without loss of generality, write in the form

$$M = \left[\begin{array}{c|c} \gamma & -\boldsymbol{u}^T \\ \hline -\,\boldsymbol{w} & \Sigma \end{array}\right]; \qquad M^{-1} = \left[\begin{array}{c|c} (\gamma - \boldsymbol{u}^T\Sigma^{-1}\boldsymbol{w})^{-1} & (\boldsymbol{u}^T/\gamma)(\Sigma - \boldsymbol{w}\boldsymbol{u}^T/\gamma)^{-1} \\ \hline (\gamma - \boldsymbol{u}^T\Sigma^{-1}\boldsymbol{w})^{-1}\Sigma^{-1}\boldsymbol{w} & (\Sigma - \boldsymbol{w}\boldsymbol{u}^T/\gamma)^{-1} \end{array}\right]. \tag{B.10}$$

Specifically, we are *not* assuming any notion of isotropy. Note that $\boldsymbol{w}\boldsymbol{u}^T = \boldsymbol{w} \otimes \boldsymbol{u}$ is a 3×3 matrix, while $\boldsymbol{u}^T\boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w}$ is a scalar. By replacing $\boldsymbol{u} \to \gamma\boldsymbol{u}$, we could also choose to write this in the completely equivalent form

$$M = \left[\begin{array}{c|c} \gamma & -\gamma\boldsymbol{u}^T \\ \hline -\,\boldsymbol{w} & \Sigma \end{array}\right]; \qquad M^{-1} = \left[\begin{array}{c|c} \gamma^{-1}(1 - \boldsymbol{u}^T\Sigma^{-1}\boldsymbol{w})^{-1} & \boldsymbol{u}^T(\Sigma - \boldsymbol{w}\boldsymbol{u}^T)^{-1} \\ \hline \gamma^{-1}(1 - \boldsymbol{u}^T\Sigma^{-1}\boldsymbol{w})^{-1}\Sigma^{-1}\boldsymbol{w} & (\Sigma - \boldsymbol{w}\boldsymbol{u}^T)^{-1} \end{array}\right]. \tag{B.11}$$

Alternatively, by now replacing $\boldsymbol{w} \to \Sigma\,\boldsymbol{w}$,

$$M = \left[\begin{array}{c|c} \gamma & -\gamma\boldsymbol{u}^T \\ \hline -\,\Sigma\boldsymbol{w} & \Sigma \end{array}\right]; \qquad M^{-1} = \left[\begin{array}{c|c} \gamma^{-1}(1 - \boldsymbol{u}^T\boldsymbol{w})^{-1} & \boldsymbol{u}^T(I - \boldsymbol{w}\boldsymbol{u}^T)^{-1}\Sigma^{-1} \\ \hline \gamma^{-1}(1 - \boldsymbol{u}^T\boldsymbol{w})^{-1}\boldsymbol{w} & (I - \boldsymbol{w}\boldsymbol{u}^T)^{-1}\Sigma^{-1} \end{array}\right]. \tag{B.12}$$

Any one of these ways of parameterizing the 4×4 matrix M is completely equivalent and mathematically acceptable, and which one we adopt is simply a matter of taste. (It is easy to explicitly carry out the matrix multiplications to verify that $MM^{-1} = I$.)

B.3.4 Aether frame and moving frame

Let us now distinguish two frames, the aether frame (the preferred rest frame) F with coordinates X, and the moving frame $\bar{F}$ with coordinates $\bar{X}$. Then for definiteness we will choose M to map from the aether frame to the moving frame, and M^{-1} to map from the moving frame to the aether frame, so that

$$\bar{X} = M\,X; \qquad X = M^{-1}\,\bar{X}. \tag{B.13}$$

(Choosing which of the frames is the aether, and which is moving, is merely a matter of convention.) We now rename things slightly and adopt the specific convention and nomenclature

$$M = \left[\begin{array}{c|c} \gamma & -\gamma\boldsymbol{u}^T \\ \hline -\,\Sigma\boldsymbol{v} & \Sigma \end{array}\right]; \qquad M^{-1} = \left[\begin{array}{c|c} \gamma^{-1}(1 - \boldsymbol{u}^T\boldsymbol{v})^{-1} & \boldsymbol{u}^T(I - \boldsymbol{v}\boldsymbol{u}^T)^{-1}\Sigma^{-1} \\ \hline \gamma^{-1}(1 - \boldsymbol{u}^T\boldsymbol{v})^{-1}\boldsymbol{v} & (I - \boldsymbol{v}\boldsymbol{u}^T)^{-1}\Sigma^{-1} \end{array}\right]. \tag{B.14}$$

Note that with these conventions both γ and Σ are dimensionless, while $\boldsymbol{v}$ has the dimensions of velocity, and $\boldsymbol{u}$ has dimensions of "slowness" $= 1/(\text{velocity})$. It is again easy to verify that $MM^{-1} = I$. It is also useful to note (see appendix B.9)

$$\boldsymbol{u}^T(I - \boldsymbol{v}\boldsymbol{u}^T)^{-1} = (1 - \boldsymbol{u}^T\boldsymbol{v})^{-1}\boldsymbol{u}^T = (1 - \boldsymbol{u}\cdot\boldsymbol{v})^{-1}\boldsymbol{u}^T, \tag{B.15}$$

and so write

$$M = \left[\begin{array}{c|c} \gamma & -\gamma\boldsymbol{u}^T \\ \hline -\,\Sigma\boldsymbol{v} & \Sigma \end{array}\right]; \qquad M^{-1} = \left[\begin{array}{c|c} \gamma^{-1}(1-\boldsymbol{u}\cdot\boldsymbol{v})^{-1} & (1-\boldsymbol{u}\cdot\boldsymbol{v})^{-1}\,\boldsymbol{u}^T\Sigma^{-1} \\ \hline \gamma^{-1}(1-\boldsymbol{u}\cdot\boldsymbol{v})^{-1}\boldsymbol{v} & (I - \boldsymbol{v}\otimes\boldsymbol{u})^{-1}\Sigma^{-1} \end{array}\right]. \tag{B.16}$$

Note that there is a kinematic singularity if $\boldsymbol{u}\cdot\boldsymbol{v} = 1$; in the particular case of special relativity this would correspond to an infinite boost to a frame traveling at lightspeed. But the possible occurrence of these kinematic singularities is a much more general phenomenon, and is not limited to special relativity. Indeed, since with our conventions

$$M = \left[\begin{array}{c|c} \gamma & \boldsymbol{0}^T \\ \hline \boldsymbol{0} & \Sigma \end{array}\right] \left[\begin{array}{c|c} 1 & -\boldsymbol{u}^T \\ \hline -\,\boldsymbol{v} & I \end{array}\right], \tag{B.17}$$

we see that

$$\det(M) = \gamma\det(\Sigma)\ \det\left[\begin{array}{c|c} 1 & -\boldsymbol{u}^T \\ \hline -\,\boldsymbol{v} & I \end{array}\right] = \gamma\det(\Sigma)\ [1 - \boldsymbol{u}\cdot\boldsymbol{v}]. \tag{B.18}$$

So the existence of the kinematic singularity is equivalent to the non-invertibility of the transformation matrix, a possibility that shoud be excluded on physical grounds.

An object that is at rest in the moving frame follows the worldline

$$\bar{X} = \begin{pmatrix} \bar{t} \\ \boldsymbol{0} \end{pmatrix}, \tag{B.19}$$

which in the aether frame coordinates maps into

$$X = M^{-1}\,\bar{X} = \bar{t}\,\gamma^{-1}(1 - \boldsymbol{u}\cdot\boldsymbol{v})^{-1}\begin{pmatrix} 1 \\ \boldsymbol{v} \end{pmatrix}. \tag{B.20}$$

That is, with these conventions the moving frame has 3-velocity $\boldsymbol{v}_{\text{moving}} = \boldsymbol{v}$ as viewed by the aether, and this is our physical interpretation of the parameter $\boldsymbol{v}$ appearing in the matrix M. But what about the aether frame as seen by the moving frame? An object at rest in the aether frame follows the worldline

$$X = \begin{pmatrix} t \\ \boldsymbol{0} \end{pmatrix}, \tag{B.21}$$

which in the moving frame coordinates maps into

$$\bar{X} = M\,X = t\begin{pmatrix} \gamma \\ -\Sigma\boldsymbol{v} \end{pmatrix}. \tag{B.22}$$

That is, as viewed in the moving frame, the aether is moving with 3-velocity

$$\boldsymbol{v}_{\text{aether}} = -\frac{\Sigma\,\boldsymbol{v}}{\gamma}. \tag{B.23}$$

Note that $\boldsymbol{v}_{\text{moving}}$ and $\boldsymbol{v}_{\text{aether}}$ are generally *not* equal-but-opposite velocities. In fact, without additional assumptions, in the general case they need not even be collinear.

B.3.5 Transformation of 3-velocity

From $\bar{X} = M\,X$ we have $d\bar{X} = M\,dX$, whence with the conventions adopted above we see

$$d\bar{t} = \gamma(dt - \boldsymbol{u}\cdot d\boldsymbol{x}); \qquad d\bar{\boldsymbol{x}} = \Sigma(d\boldsymbol{x} - \boldsymbol{v}dt), \tag{B.24}$$

so that

$$\dot{\bar{\boldsymbol{x}}} \equiv \frac{d\bar{\boldsymbol{x}}}{d\bar{t}} = \frac{\Sigma(\dot{\boldsymbol{x}} - \boldsymbol{v})}{\gamma(1 - \boldsymbol{u}\cdot\dot{\boldsymbol{x}})}\;. \tag{B.25}$$

This is the general combination of velocities rule. (One can easily see that it is a natural generalization of the usual special relativistic combination of velocities, with current conventions being chosen to make this transformation as simple as possible). Note in particular that an object at rest in the aether frame, with $\dot{\boldsymbol{x}} = \boldsymbol{0}$, moves at 3-velocity $-\Sigma\boldsymbol{v}/\gamma$ in the moving frame, while an object at rest in the moving frame, with $\dot{\bar{\boldsymbol{x}}} = \boldsymbol{0}$, moves at 3-velocity $\boldsymbol{v}$ in the aether frame.

Similarly, from $dX = M^{-1}\,d\bar{X}$ we have

$$dt = \gamma^{-1}(1 - \boldsymbol{u}\cdot\boldsymbol{v})^{-1}d\bar{t} + (1 - \boldsymbol{u}\cdot\boldsymbol{v})^{-1}\boldsymbol{u}^T\Sigma^{-1}d\bar{\boldsymbol{x}}, \tag{B.26}$$

and

$$d\boldsymbol{x} = (I - \boldsymbol{v}\otimes\boldsymbol{u})^{-1}\Sigma^{-1}d\bar{\boldsymbol{x}} + \gamma^{-1}(1 - \boldsymbol{u}\cdot\boldsymbol{v})^{-1}\boldsymbol{v}d\bar{t}, \tag{B.27}$$

so that

$$\dot{\boldsymbol{x}} \equiv \frac{d\boldsymbol{x}}{dt} = \frac{(I - \boldsymbol{v}\otimes\boldsymbol{u})^{-1}\Sigma^{-1}\dot{\bar{\boldsymbol{x}}} + \gamma^{-1}(1 - \boldsymbol{u}\cdot\boldsymbol{v})^{-1}\boldsymbol{v}}{\gamma^{-1}(1 - \boldsymbol{u}\cdot\boldsymbol{v})^{-1} + (1 - \boldsymbol{u}\cdot\boldsymbol{v})^{-1}\boldsymbol{u}^T\Sigma^{-1}\dot{\bar{\boldsymbol{x}}}}\;. \tag{B.28}$$

We can simplify this to obtain

$$\dot{\boldsymbol{x}} = \frac{\gamma(1 - \boldsymbol{u}\cdot\boldsymbol{v})(I - \boldsymbol{v}\otimes\boldsymbol{u})^{-1}\Sigma^{-1}\dot{\bar{\boldsymbol{x}}} + \boldsymbol{v}}{1 + \gamma\boldsymbol{u}^T\Sigma^{-1}\dot{\bar{\boldsymbol{x}}}}. \tag{B.29}$$

But (see appendix B.9)

$$(1 - \boldsymbol{u}\cdot\boldsymbol{v})(I - \boldsymbol{v}\otimes\boldsymbol{u})^{-1} = (1 - \boldsymbol{u}\cdot\boldsymbol{v})I + \boldsymbol{v}\otimes\boldsymbol{u}, \tag{B.30}$$

and so

$$\dot{\boldsymbol{x}} = \frac{\gamma[(1 - \boldsymbol{u}\cdot\boldsymbol{v})I + \boldsymbol{v}\otimes\boldsymbol{u}]\,\Sigma^{-1}\dot{\bar{\boldsymbol{x}}} + \boldsymbol{v}}{1 + \gamma\boldsymbol{u}^T\Sigma^{-1}\dot{\bar{\boldsymbol{x}}}.}. \tag{B.31}$$

Finally

$$\dot{\boldsymbol{x}} = \frac{\gamma(1 - \boldsymbol{u}\cdot\boldsymbol{v})\,\Sigma^{-1}\dot{\bar{\boldsymbol{x}}}}{1 + \gamma \boldsymbol{u}^T \Sigma^{-1}\dot{\bar{\boldsymbol{x}}}} + \boldsymbol{v}. \tag{B.32}$$

This last formula seems to be the best one can do. Attempting to change conventions to simplify *this* particular formula leads to problems elsewhere. Note in particular that something at rest in the moving frame, so that $\dot{\bar{\boldsymbol{x}}} = 0$, moves at velocity $\boldsymbol{v}$ in the aether frame.

B.3.6 Groupoid/pseudogroup structure

Physically the matrix M need not be a function of $\boldsymbol{v}$ only — it can also depend on the orientation of the moving inertial frame with respect to the preferred frame, and worse the quantities γ, $\boldsymbol{u}$, and Σ, are (potentially) free parameters in their own right. It is useful to write $M(\bar{F})$, to emphasise that the matrix $M(\bar{F})$ is potentially a function of all the parameters characterizing the moving inertial frame $\bar{F}$. (We could furthermore write the various pieces of $M(\bar{F})$ as $\gamma(\bar{F})$, $\boldsymbol{v}(\bar{F})$, $\boldsymbol{u}(\bar{F})$, and $\Sigma(\bar{F})$; while technically more correct, this is so clumsy as to be impracticable, and the frame dependence of these quantities will always be implicitly understood.) In addition, one should keep in mind that in general the transformation matrices $M(F)$ could also depend on the particular type of rulers and clocks one is using; it is only for situations of very high symmetry — essentially amounting to adoption of the relativity principle — that the notion of time and distance can be abstracted to have a meaning that is independent of the internal structure of one's choice of clocks and rulers.

Note that in general the set $\{M(\bar{F})\}$, (where $M(\bar{F})$ is the transformation matrix from the aether inertial frame to the moving inertial frame $\bar{F}$), need not form a group, similarly the set $\{M^{-1}(\bar{F})\}$ need not form a group. There is no need for these sets to be closed under matrix multiplication. Nor are these sets generally closed under matrix inversion. There does not seem to be any specialized mathematical terminology for such objects — they are not semigroups, they are not groupoids, they are not pseudogroups, they are not monoids, they are not cosets, they are not magmas, they are just sets of matrices.

To transform from an arbitrary inertial frame F_1 to another arbitrary inertial frame F_2, the appropriate transformation is

$$M(F_2, F_1) = M(F_2)\,M(F_1)^{-1}. \tag{B.33}$$

The set $\{M(F_2, F_1)\} = \{M(F_2)\,M(F_1)^{-1}\}$ certainly forms a *groupoid/pseudogroup*, in the sense that the set is closed under the restricted set of compositions (so-called *partial-products*) of the form

$$M(F_3, F_2) M(F_2, F_1) = M(F_3, F_1). \tag{B.34}$$

(The relevant mathematical terminology is not 100% standardized, and different sources prefer to call this either a *groupoid* or a *pseudogroup*.) Note that $M(F, F) = I$, so an identity certainly exists, and that $M(F_2, F_1)^{-1} = M(F_1, F_2)$ so that inverses also exist. Associativity is automatic because matrix multiplication is associative. In general this is the most you can say. The technical difference between a group and a *groupoid/pseudogroup* is in this context *extremely important*. It is this technical mathematical distinction that ultimately allows us to side-step the usual von Ignatowsky theorems (and their variants) that under normal circumstances lead almost uniquely to the Lorentz group or the Galileo group — the physics reason for this extra generality is because while we have assumed linearity of the transformations we have not assumed either isotropy or the relativity principle.

B.4 Transformations of energy and momentum

B.4.1 Defining energy and momentum

Defining energy and momentum, as opposed to purely kinematical notions of velocity and position, requires at least some notion of dynamics. Pick some arbitrary but fixed inertial frame. Ignoring interactions for now, in view of the homogeneity of spacetime we shall assume that each particle has associated with it some Lagrangian $L(\dot{\boldsymbol{x}})$ which leads to a momentum $\boldsymbol{p}(\dot{\boldsymbol{x}}) = \partial L/\partial\dot{\boldsymbol{x}}$, and hence to a Hamiltonian $H(\boldsymbol{p})$, which we shall typically just write as $E(\boldsymbol{p})$. Because of space-time homogeneity and the Hamiltonian/Lagrangian framework, Noether's theorem implies energy and momentum conservation:

$$\sum_{\text{in}} E_i = \sum_{\text{out}} E_i; \qquad\qquad \sum_{\text{in}} \boldsymbol{p}_i = \sum_{\text{out}} \boldsymbol{p}_i. \tag{B.35}$$

Now the inertial equations $\ddot{\boldsymbol{x}} = 0$ will be satisfied for any arbitrary $L(\dot{\boldsymbol{x}})$. (Note the absence of any explicit t or $\boldsymbol{x}$ dependence.) To operationally determine a specific $L(\dot{\boldsymbol{x}})$ that can usefully characterize a specific particle, you will have to perform a large number of collisions at various 3-velocities, compare input and output states, and data-fit to extract suitable $E_i(\dot{\boldsymbol{x}})$ and $\boldsymbol{p}_i(\dot{\boldsymbol{x}})$ corresponding to the various particles in your universe of discourse. Once this is done you can build a model for the $L_i(\dot{\boldsymbol{x}})$ using

$$L_i(\dot{\boldsymbol{x}}) = L_i(\boldsymbol{0}) + \int_{\boldsymbol{0}}^{\dot{\boldsymbol{x}}} \boldsymbol{p}(\dot{\tilde{\boldsymbol{x}}}) \cdot d\dot{\tilde{\boldsymbol{x}}}. \tag{B.36}$$

Note that, in modelling the $\boldsymbol{p}_i(\dot{\boldsymbol{x}})$, one would have to take into account the consistency condition $\nabla_{\dot{\boldsymbol{x}}} \times \boldsymbol{p}_i(\dot{\boldsymbol{x}})$ required for this construction to be path independent in velocity space.

But even after one has done this, the construction *cannot be unique* — for any set of *constants* ϵ_i and $\boldsymbol{\pi}_i$ such that $\sum_{\text{in}} \epsilon_i = \sum_{\text{out}} \epsilon_i$ and $\sum_{\text{in}} \boldsymbol{\pi}_i = \sum_{\text{out}} \boldsymbol{\pi}_i$ we see that the assignments

$$E_i \leftrightarrow E_i + \epsilon_i; \qquad \boldsymbol{p}_i \leftrightarrow \boldsymbol{p}_i + \boldsymbol{\pi}_i \tag{B.37}$$

are physically indistinguishable. But that means the Lagrangians

$$L_i(\dot{\boldsymbol{x}}) \leftrightarrow L_i(\dot{\boldsymbol{x}}) - \epsilon_i + \boldsymbol{\pi}_i \cdot \dot{\boldsymbol{x}} \tag{B.38}$$

are physically indistinguishable. In terms of the action this means

$$S_i = \int L_i(\dot{\boldsymbol{x}})dt \leftrightarrow S_i = \int L_i(\dot{\boldsymbol{x}})dt - \epsilon_i(t_F - t_I) + \boldsymbol{\pi}_i \cdot (\boldsymbol{x}_F - \boldsymbol{x}_I) \tag{B.39}$$

are physically indistinguishable — which is physically and mathematically obvious in view of the fact that the two actions differ only by boundary terms. This intrinsic ambiguity in the definition of energy and momentum will (perhaps unfortunately) turn out to be important. One could try to resolve these ambiguities in a number of different ways:

- For instance, the ambiguity in momentum could be fixed by setting the momentum at zero velocity to be zero: $\boldsymbol{p}(\dot{\boldsymbol{x}} = \boldsymbol{0}) = \boldsymbol{0}$. Sometimes this works well, sometimes it does not.
- The ambiguity in energy is equivalent to an ambiguity in rest energy $E(\dot{\boldsymbol{x}} = \boldsymbol{0})$; attempting to set the rest energy to zero is often severely problematic.

In general it is best to keep this freedom available in the calculation as long as possible.

B.4.2 Affine versus linear transformations

What can we now say about energy and momentum, and their transformation properties, using only linearity of the transformations between inertial frames? (Recall that we very specifically do not assume isotropy or any form of the relativity principle.)

Consider a single particle, but multiple inertial frames. To even begin to talk about energy and momentum, in each frame one must be able to set up a suitable Lagrangian and Hamiltonian, and there should be some as yet unspecified relationship between the Lagrangians and Hamiltonians in these distinct inertial frames. Furthermore extrema of the action as calculated in one inertial frame must coincide with extrema of the action calculated in any other inertial frame.

That is, in complete generality we should demand that for any two inertial frames the action calculated in these frames should be equal up to boundary terms, and in each frame we know the action is ambiguous up to boundary terms. In view of the groupoid structure of the transformations between inertial frames there is no loss of generality in considering one moving frame $\bar{F}$ plus the aether frame F for which we can write

$$\int \bar{L}\, d\bar{t} + (\text{boundary terms}) = \int L\, dt + (\text{boundary terms}). \tag{B.40}$$

In view of our previous discussion this implies

$$\int \left\{ \bar{L} - \bar{\epsilon} + \bar{\boldsymbol{\pi}} \cdot (d\bar{\boldsymbol{x}}/d\bar{t}) \right\} d\bar{t} = \int \{ L - \epsilon + \boldsymbol{\pi} \cdot (d\boldsymbol{x}/dt) \} \, dt. \tag{B.41}$$

But since $L = -(E - \boldsymbol{p} \cdot \dot{\boldsymbol{x}})$ this implies

$$\int \left\{ (\bar{E} + \bar{\epsilon}) - (\bar{\boldsymbol{p}} + \bar{\boldsymbol{\pi}}) \cdot (d\bar{\boldsymbol{x}}/d\bar{t}) \right\} d\bar{t} = \int \{ (E + \epsilon) - (\boldsymbol{p} + \boldsymbol{\pi}) \cdot (d\boldsymbol{x}/dt) \} \, dt. \tag{B.42}$$

Therefore

$$(\bar{E} + \bar{\epsilon}) \, d\bar{t} - (\bar{\boldsymbol{p}} + \bar{\boldsymbol{\pi}}) \cdot d\bar{\boldsymbol{x}} = (E + \epsilon) \, dt - (\boldsymbol{p} + \boldsymbol{\pi}) \cdot d\boldsymbol{x}. \tag{B.43}$$

So now in complete generality we have

$$(\bar{E} + \bar{\epsilon}, -\bar{\boldsymbol{p}}^T - \bar{\boldsymbol{\pi}}^T) \begin{pmatrix} d\bar{t} \\ d\bar{\boldsymbol{x}} \end{pmatrix} = (E + \epsilon, -\boldsymbol{p}^T - \boldsymbol{\pi}^T) \begin{pmatrix} dt \\ d\boldsymbol{x} \end{pmatrix}. \tag{B.44}$$

But

$$\begin{pmatrix} d\bar{t} \\ d\bar{\boldsymbol{x}} \end{pmatrix} = M \begin{pmatrix} dt \\ d\boldsymbol{x} \end{pmatrix}, \tag{B.45}$$

therefore implying both

$$(E + \epsilon, -\boldsymbol{p}^T - \boldsymbol{\pi}^T) = (\bar{E} + \bar{\epsilon}, -\bar{\boldsymbol{p}}^T - \bar{\boldsymbol{\pi}}^T) \, M, \tag{B.46}$$

and

$$(\bar{E} + \bar{\epsilon}, -\bar{\boldsymbol{p}}^T - \bar{\boldsymbol{\pi}}^T) = (E + \epsilon, -\boldsymbol{p}^T - \boldsymbol{\pi}^T) \, M^{-1}. \tag{B.47}$$

These are *affine* transformation laws for energy and momentum, (that is, linear plus an inhomogeneous offset), with the affine piece only depending on the intrinsic ambiguities $\left(\epsilon, -\boldsymbol{\pi}^T\right)$ and $\left(\bar{\epsilon}, -\bar{\boldsymbol{\pi}}^T\right)$ in the energy and momentum. Note that $P = (E, -\boldsymbol{p}^T)$ transforms in the dual space [that is, dual to 4-position $X = (t, \boldsymbol{x}^T)^T$]. To be explicit about this

$$E \to \bar{E} = \frac{E - \boldsymbol{p}^T \boldsymbol{v}}{\gamma(1 - \boldsymbol{u}^T \boldsymbol{v})} + \frac{\epsilon - \boldsymbol{\pi}^T \boldsymbol{v}}{\gamma(1 - \boldsymbol{u}^T \boldsymbol{v})} - \bar{\epsilon}, \tag{B.48}$$

and

$$\boldsymbol{p} \to \bar{\boldsymbol{p}} = (\Sigma^{-1})^T (I - \boldsymbol{u}\boldsymbol{v}^T)^{-1} (\boldsymbol{p} - E\boldsymbol{u}) + (\Sigma^{-1})^T (I - \boldsymbol{u}\boldsymbol{v}^T)^{-1} (\boldsymbol{\pi} - \epsilon \boldsymbol{u}) - \bar{\boldsymbol{\pi}}. \tag{B.49}$$

In terms of dot and tensor products we can rewrite this as

$$E \to \bar{E} = \frac{E - \boldsymbol{p} \cdot \boldsymbol{v}}{\gamma(1 - \boldsymbol{u} \cdot \boldsymbol{v})} + \frac{\epsilon - \boldsymbol{\pi} \cdot \boldsymbol{v}}{\gamma(1 - \boldsymbol{u} \cdot \boldsymbol{v})} - \bar{\epsilon}, \tag{B.50}$$

and

$$\boldsymbol{p} \to \bar{\boldsymbol{p}} = (\Sigma^{-1})^T (I - \boldsymbol{u} \otimes \boldsymbol{v})^{-1} (\boldsymbol{p} - E\boldsymbol{u}) + (\Sigma^{-1})^T (I - \boldsymbol{u} \otimes \boldsymbol{v})^{-1} (\boldsymbol{\pi} - \epsilon \boldsymbol{u}) - \bar{\boldsymbol{\pi}}. \tag{B.51}$$

(One can now begin to see how the Lorentz and Galilean transformations might emerge as special cases of this very general result.) The inverse transformations are somewhat simpler

$$\bar{E} \to E = \gamma \bar{E} + \bar{\boldsymbol{p}}^T \Sigma \boldsymbol{v} + \gamma \bar{\epsilon} + \bar{\boldsymbol{\pi}}^T \Sigma \boldsymbol{v} - \epsilon, \tag{B.52}$$

and

$$\bar{\boldsymbol{p}} \to \boldsymbol{p} = \gamma \bar{E} \boldsymbol{u} + \Sigma^T \bar{\boldsymbol{p}} + \gamma \bar{\epsilon} \boldsymbol{u} + \Sigma^T \bar{\boldsymbol{\pi}} - \boldsymbol{\pi}. \tag{B.53}$$

Suppose we now consider the *same* particle at two different 3-velocities, but working with the same two inertial frmaes F and $\bar{F}$; then in terms of energy and momentum *differences*, we can write a *homogeneous* linear transformation law of the form

$$\left([\bar{E}_1 - \bar{E}_2], -[\bar{\boldsymbol{p}}_1 - \bar{\boldsymbol{p}}_2]^T\right) = \left([E_1 - E_2], -[\boldsymbol{p}_1 - \boldsymbol{p}_2]^T\right)\ M^{-1}. \tag{B.54}$$

That is:

$$\Delta E \to \Delta \bar{E} = \frac{\Delta E - \Delta \boldsymbol{p} \cdot \boldsymbol{v}}{\gamma(1 - \boldsymbol{u} \cdot \boldsymbol{v})}, \tag{B.55}$$

and

$$\Delta \boldsymbol{p} \to \Delta \bar{\boldsymbol{p}} = (\Sigma^{-1})^T (I - \boldsymbol{u} \otimes \boldsymbol{v})^{-1} (\Delta \boldsymbol{p} - \Delta E \boldsymbol{u}). \tag{B.56}$$

We need to compare the *same* particle at two different velocities, since otherwise there is no particular reason for the $\left(\epsilon, -\boldsymbol{\pi}^T\right)$ and $\left(\bar{\epsilon}, -\bar{\boldsymbol{\pi}}^T\right)$ to be the same for the two situations. Note that for two otherwise identical particles one could in principle choose differing values for the parameters $\left(\epsilon, -\boldsymbol{\pi}^T\right)$ and $\left(\bar{\epsilon}, -\bar{\boldsymbol{\pi}}^T\right)$, thereby making them distinguishable. This does not appear to be what happens in our universe, so we shall assume that the quantities $\left(\epsilon, -\boldsymbol{\pi}^T\right)$ and $\left(\bar{\epsilon}, -\bar{\boldsymbol{\pi}}^T\right)$, while they might depend on the inertial frame one is working in, are at least universal for particular particle species.

Note that in terms of energy-momentum differences the inverse transformations are

$$\Delta \bar{E} \to \Delta E = \gamma \Delta \bar{E} + \Delta \bar{\boldsymbol{p}}^T \Sigma \boldsymbol{v}, \tag{B.57}$$

and

$$\Delta \bar{\boldsymbol{p}} \to \Delta \boldsymbol{p} = \gamma \Delta \bar{E} \boldsymbol{u} + \Sigma^T \, \Delta \bar{\boldsymbol{p}}. \tag{B.58}$$

As a consistency check on the general formalism we can readily verify that these energy-momentum transformation laws are compatible with, and permit us to recover, the purely kinematical velocity combination rules. See appendix B.10 for details.

B.4.3 Summary

For each individual particle species we have

$$E \to \bar{E} = \frac{E - \boldsymbol{p} \cdot \boldsymbol{v}}{\gamma(1 - \boldsymbol{u} \cdot \boldsymbol{v})} + \frac{\epsilon - \boldsymbol{\pi} \cdot \boldsymbol{v}}{\gamma(1 - \boldsymbol{u} \cdot \boldsymbol{v})} - \bar{\epsilon}, \tag{B.59}$$

and

$$\boldsymbol{p} \to \bar{\boldsymbol{p}} = (\Sigma^{-1})^T (I - \boldsymbol{u} \otimes \boldsymbol{v})^{-1} (\boldsymbol{p} - E\boldsymbol{u}) + (\Sigma^{-1})^T (I - \boldsymbol{u} \otimes \boldsymbol{v})^{-1} (\boldsymbol{\pi} - \epsilon \boldsymbol{u}) - \bar{\boldsymbol{\pi}}, \quad \text{(B.60)}$$

while the inverse transformations are

$$\bar{E} \to E = -\epsilon + \gamma \bar{E} + \bar{\boldsymbol{p}} \cdot (\Sigma \boldsymbol{v}) + \gamma \bar{\epsilon} + \bar{\boldsymbol{\pi}} \cdot (\Sigma \boldsymbol{v}), \quad \text{(B.61)}$$

and

$$\bar{\boldsymbol{p}} \to \boldsymbol{p} = -\boldsymbol{\pi} + \gamma \bar{E} \boldsymbol{u} + \Sigma^T \bar{\boldsymbol{p}} + \gamma \bar{\epsilon} \boldsymbol{u} + \Sigma^T \bar{\boldsymbol{\pi}}. \quad \text{(B.62)}$$

We have a certain amount of freedom to choose ϵ and $\boldsymbol{\pi}$, and $\bar{\epsilon}$ and $\bar{\boldsymbol{\pi}}$. One obvious choice would be to always make the transformation laws linear; however as we shall soon see this is not always the best thing to do.

B.5 Examples

Let us now consider the very standard cases of Galilean invariance and Lorentz invariance, comparing affine and linear transformation laws for energy-momentum.

B.5.1 Galileo group (affine version)

For Galilean kinematics we have

$$M = \left[\begin{array}{c|c} 1 & \mathbf{0}^T \\ \hline -\boldsymbol{v} & I \end{array} \right], \quad \text{(B.63)}$$

so

$$\bar{t} = t; \qquad \bar{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{v} t; \qquad \dot{\bar{\boldsymbol{x}}} = \dot{\boldsymbol{x}} - \boldsymbol{v}. \quad \text{(B.64)}$$

Now *one* natural choice is to choose the particularly simple Lagrangians

$$L = \frac{1}{2} m ||\dot{\boldsymbol{x}}||^2; \qquad \bar{L} = \frac{1}{2} m ||\dot{\bar{\boldsymbol{x}}}||^2. \quad \text{(B.65)}$$

(We shall soon see that there are also other choices one can make.) Then

$$\bar{L} = \frac{1}{2} m ||\dot{\bar{\boldsymbol{x}}}||^2 = \frac{1}{2} m ||\dot{\boldsymbol{x}} - \boldsymbol{v}||^2 = \frac{1}{2} m ||\dot{\boldsymbol{x}}||^2 - m \boldsymbol{v} \cdot \dot{\boldsymbol{x}} + \frac{1}{2} m ||\boldsymbol{v}||^2. \quad \text{(B.66)}$$

That is

$$\bar{L} = L + \frac{1}{2} m ||\boldsymbol{v}||^2 - m \boldsymbol{v} \cdot \dot{\boldsymbol{x}}. \quad \text{(B.67)}$$

Now note

$$\bar{\boldsymbol{p}} = m\dot{\boldsymbol{x}} - m\boldsymbol{v} = m\dot{\bar{\boldsymbol{x}}}; \qquad \bar{H} = \bar{\boldsymbol{p}} \cdot \dot{\bar{\boldsymbol{x}}} - \bar{L} = \frac{||\bar{\boldsymbol{p}}||^2}{2m}. \tag{B.68}$$

So working explicitly, with these particular conventions, we have *affine* transformations for energy-momentum:

$$\bar{E} = E - \boldsymbol{p} \cdot \boldsymbol{v} + \frac{1}{2} m ||\boldsymbol{v}||^2; \qquad \bar{\boldsymbol{p}} = \boldsymbol{p} - m\boldsymbol{v}. \tag{B.69}$$

In contrast, working directly from the general transformation laws derived above, and taking $\gamma = 1$, $\boldsymbol{u} = 0$, and $\Sigma = I$, we have

$$E \to \bar{E} = -\bar{\epsilon} + E - \boldsymbol{p} \cdot \boldsymbol{v} + [\epsilon - \boldsymbol{\pi} \cdot \boldsymbol{v}] = E - \boldsymbol{p} \cdot \boldsymbol{v} + \frac{1}{2} m ||\boldsymbol{v}||^2, \tag{B.70}$$

and

$$\boldsymbol{p} \to \bar{\boldsymbol{p}} = -\bar{\boldsymbol{\pi}} + \boldsymbol{p} + \boldsymbol{\pi} = \boldsymbol{p} - m\boldsymbol{v}, \tag{B.71}$$

from which we deduce that this particular way of implementing Galilean mechanics corresponds to the choices

$$\bar{\epsilon} = -\frac{1}{2} m ||\boldsymbol{v}||^2; \qquad \bar{\boldsymbol{\pi}} = m\boldsymbol{v}; \qquad \epsilon = 0; \qquad \boldsymbol{\pi} = 0. \tag{B.72}$$

(Remember that by convention $\bar{F}$ is the moving frame while F is the "aether" frame. Note that it is the quantities $\{\bar{\epsilon}, \bar{\boldsymbol{\pi}}\}$ associated with the moving frame that are non-zero, and that these quantities depend on the velocity $\boldsymbol{v}$ of the moving frame.) The inverse transformations are

$$\bar{E} \to E = -\epsilon + \bar{E} + \bar{\boldsymbol{p}} \cdot \boldsymbol{v} + \bar{\epsilon} + \bar{\boldsymbol{\pi}} \cdot \boldsymbol{v} = \bar{E} + \bar{\boldsymbol{p}} \cdot \boldsymbol{v} + \frac{1}{2} m ||\boldsymbol{v}||^2, \tag{B.73}$$

and

$$\bar{\boldsymbol{p}} \to \boldsymbol{p} = -\boldsymbol{\pi} + \bar{\boldsymbol{p}} + \bar{\boldsymbol{\pi}} = \bar{\boldsymbol{p}} + m\boldsymbol{v}. \tag{B.74}$$

This is the "usual" way of doing Galilean dynamics, which unavoidably leads to *affine* transformations for energy and momentum.

A somewhat subtle message to be taken from the discussion is this: Since affine transformations arise so naturally in this extremely straightforward setting, it seems unlikely that the affine features of the energy-momentum transformations could always be completely eliminated in more general settings.

B.5.2 Lorentz group (linear version)

In this case the Lorentz transformations are

$$M = \left[\begin{array}{c|c} \gamma & -\gamma \boldsymbol{v}^T / c^2 \\ \hline -\gamma \boldsymbol{v} & \gamma \boldsymbol{n} \otimes \boldsymbol{n} + [I - \boldsymbol{n} \otimes \boldsymbol{n}] \end{array} \right] \tag{B.75}$$

with $\boldsymbol{v} = v\boldsymbol{n}$ and $\gamma = (1 - v^2/c^2)^{-1/2}$. The *usual* form of the Lagrangian is

$$L = -mc^2\sqrt{1 - ||\dot{\boldsymbol{x}}||^2/c^2}, \tag{B.76}$$

so

$$\boldsymbol{p} = \frac{m\dot{\boldsymbol{x}}}{\sqrt{1 - ||\dot{\boldsymbol{x}}||^2/c^2}}; \qquad H = \boldsymbol{p}\cdot\dot{\boldsymbol{x}} - L = \frac{mc^2}{\sqrt{1 - ||\dot{\boldsymbol{x}}||^2/c^2}}. \tag{B.77}$$

Furthermore

$$\bar{L} = -mc^2\sqrt{1 - ||\dot{\bar{\boldsymbol{x}}}||^2/c^2}, \tag{B.78}$$

so

$$\bar{\boldsymbol{p}} = \frac{m\dot{\bar{\boldsymbol{x}}}}{\sqrt{1 - ||\dot{\bar{\boldsymbol{x}}}||^2/c^2}}; \qquad \bar{H} = \bar{\boldsymbol{p}}\cdot\dot{\bar{\boldsymbol{x}}} - \bar{L} = \frac{mc^2}{\sqrt{1 - ||\dot{\bar{\boldsymbol{x}}}||^2/c^2}}, \tag{B.79}$$

and in fact

$$\bar{L}\; d\bar{t} = L\; dt, \tag{B.80}$$

implying both $\epsilon = 0$ and $\boldsymbol{\pi} = \mathbf{0}$, and $\bar{\epsilon} = 0$ and $\bar{\boldsymbol{\pi}} = \mathbf{0}$. Then the energy-momentum transformations are just the usual *linear* Lorentz transformations

$$(E, -\boldsymbol{p}^T) = (\bar{E}, -\bar{\boldsymbol{p}}^T)M, \tag{B.81}$$

and

$$(\bar{E}, -\bar{\boldsymbol{p}}^T) = (E, -\boldsymbol{p}^T)M^{-1}. \tag{B.82}$$

This is the standard way of implementing Lagrangian and Hamiltonian mechanics in the presence of Lorentz symmetry.

B.5.3 Lorentz group (affine version)

We could have chosen a slightly different normalization for L and H. If we take

$$L = mc^2\left\{1 - \sqrt{1 - ||\dot{\boldsymbol{x}}||^2/c^2}\right\}, \tag{B.83}$$

then

$$\boldsymbol{p} = \frac{m\dot{\boldsymbol{x}}}{\sqrt{1 - ||\dot{\boldsymbol{x}}||^2/c^2}}; \qquad H = \boldsymbol{p}\cdot\dot{\boldsymbol{x}} - L = \frac{mc^2}{\sqrt{1 - ||\dot{\boldsymbol{x}}||^2/c^2}} - mc^2. \tag{B.84}$$

Furthermore

$$\bar{L} = mc^2\left\{1 - \sqrt{1 - ||\dot{\bar{\boldsymbol{x}}}||^2/c^2}\right\}, \tag{B.85}$$

so

$$\bar{\boldsymbol{p}} = \frac{m\dot{\bar{\boldsymbol{x}}}}{\sqrt{1 - ||\dot{\bar{\boldsymbol{x}}}||^2/c^2}}; \qquad \bar{H} = \bar{\boldsymbol{p}}\cdot\dot{\bar{\boldsymbol{x}}} - \bar{L} = \frac{mc^2}{\sqrt{1 - ||\dot{\bar{\boldsymbol{x}}}||^2/c^2}} - mc^2. \tag{B.86}$$

In fact with this normalization

$$[\bar{L} - mc^2]\, d\bar{t} = [L - mc^2]\, dt, \tag{B.87}$$

whence

$$\bar{\epsilon} = mc^2; \quad \bar{\boldsymbol{\pi}} = \mathbf{0}; \qquad \text{and} \qquad \epsilon = mc^2; \qquad \boldsymbol{\pi} = \mathbf{0}. \tag{B.88}$$

We can rephrase this in terms of the 4-velocities of the "aether" and moving frames as

$$\begin{pmatrix} \bar{\epsilon} \\ \bar{\boldsymbol{\pi}} \end{pmatrix} = mc^2\, \bar{V}; \qquad \begin{pmatrix} \epsilon \\ \boldsymbol{\pi} \end{pmatrix} = mc^2\, V; \tag{B.89}$$

With these choices the energy-momentum transformations look slightly unusual. Taking $\boldsymbol{v} \parallel \boldsymbol{p}$ for simplicity (the non-collinear case does not teach us anything new) we now have

$$E \to \bar{E} = \gamma([mc^2 + E] - \boldsymbol{p} \cdot \boldsymbol{v}) - mc^2, \tag{B.90}$$

and

$$\boldsymbol{p} \to \bar{\boldsymbol{p}} = \gamma(\boldsymbol{p} - [mc^2 + E]\boldsymbol{v}/c^2). \tag{B.91}$$

The inverse transformations are

$$\bar{E} \to E = \gamma([mc^2 + \bar{E}] + \bar{\boldsymbol{p}} \cdot \boldsymbol{v}) - mc^2, \tag{B.92}$$

and

$$\bar{\boldsymbol{p}} \to \boldsymbol{p} = \gamma([mc^2 + \bar{E}]\boldsymbol{v}/c^2 + \bar{\boldsymbol{p}}). \tag{B.93}$$

These *affine* transformations make perfectly good physical sense once you realize that, with the conventions of this subsection, the E's in question are just the relativistic kinetic energies — what is normally denoted by K:

$$E_{\text{here}} = E_{\text{total}} - mc^2 = K. \tag{B.94}$$

Then

$$K \to \bar{K} = \gamma([mc^2 + K] - \boldsymbol{p} \cdot \boldsymbol{v}) - mc^2, \tag{B.95}$$

and

$$\boldsymbol{p} \to \bar{\boldsymbol{p}} = \gamma(\boldsymbol{p} - [mc^2 + K]\boldsymbol{v}/c^2), \tag{B.96}$$

while

$$\bar{K} \to K = \gamma([mc^2 + \bar{K}] + \bar{\boldsymbol{p}} \cdot \boldsymbol{v}) - mc^2, \tag{B.97}$$

and

$$\bar{\boldsymbol{p}} \to \boldsymbol{p} = \gamma([mc^2 + \bar{K}]\boldsymbol{v}/c^2 + \bar{\boldsymbol{p}}). \tag{B.98}$$

These are manifestly just a disguised form of the usual Lorentz transformations. Note that the formal $c \to \infty$ limit of *these* (slightly nonstandard) affine equations is

$$K \to \bar{K} = K - \boldsymbol{p} \cdot \boldsymbol{v} + \frac{1}{2} m ||\boldsymbol{v}||^2; \qquad \boldsymbol{p} \to \bar{\boldsymbol{p}} = \boldsymbol{p} - m\boldsymbol{v}; \tag{B.99}$$

and

$$\bar{K} \to K = \bar{K} + \bar{\boldsymbol{p}} \cdot \boldsymbol{v} + \frac{1}{2} m ||\boldsymbol{v}||^2; \qquad \bar{\boldsymbol{p}} \to \boldsymbol{p} = \bar{\boldsymbol{p}} + m\boldsymbol{v}. \tag{B.100}$$

These are the transformation laws for (the usual form of) the Galileo group.

Again, the somewhat subtle message to take from this is that since the affine parameters ϵ and $\boldsymbol{\pi}$, and $\bar{\epsilon}$ and $\bar{\boldsymbol{\pi}}$, are already so important in situations of extremely high symmetry (the Lorentz group, the Galileo group), then they are also likely to be important in any situations where these symmetries are broken.

B.5.4 Galileo group (linear version)

The previous discussion suggests that there might be *some* (perhaps nonstandard) set of conventions that would make the energy and momentum transform linearly for the Galileo group. That is, there might be *some* way of arranging things so that for the Galileo group

$$E \to \bar{E} = E - \boldsymbol{p} \cdot \boldsymbol{v}; \qquad \boldsymbol{p} \to \bar{\boldsymbol{p}} = \boldsymbol{p}. \tag{B.101}$$

How would we do that? It will have to be something rather unusual. Choose the following Lagrangians:

$$L = \frac{1}{2} m ||\dot{\boldsymbol{x}}||^2; \qquad \bar{L} = \frac{1}{2} m ||\dot{\bar{\boldsymbol{x}}} + \boldsymbol{v}||^2. \tag{B.102}$$

Then the momenta are

$$\boldsymbol{p} = m\dot{\boldsymbol{x}}; \qquad \bar{\boldsymbol{p}} = m(\dot{\bar{\boldsymbol{x}}} + \boldsymbol{v}) = m\dot{\boldsymbol{x}} = \boldsymbol{p}. \tag{B.103}$$

The energy in the aether frame is (as usual)

$$E = \boldsymbol{p} \cdot \dot{\boldsymbol{x}} - L = \frac{1}{2} m ||\dot{\boldsymbol{x}}||^2. \tag{B.104}$$

However with these conventions the energy in the moving frame is

$$\begin{aligned}
\bar{E} = \bar{\boldsymbol{p}} \cdot \dot{\bar{\boldsymbol{x}}} - \bar{L} &= m(\dot{\bar{\boldsymbol{x}}} + \boldsymbol{v}) \cdot \dot{\bar{\boldsymbol{x}}} - \frac{1}{2} m ||\dot{\bar{\boldsymbol{x}}} + \boldsymbol{v}||^2 && \text{(B.105)}\\
&= m\dot{\boldsymbol{x}} \cdot (\dot{\boldsymbol{x}} - \boldsymbol{v}) - \frac{1}{2} m ||\dot{\boldsymbol{x}}||^2 && \text{(B.106)}\\
&= \frac{1}{2} m ||\dot{\boldsymbol{x}}||^2 - m\dot{\boldsymbol{x}} \cdot \boldsymbol{v} = E - \boldsymbol{p} \cdot \boldsymbol{v}. && \text{(B.107)}
\end{aligned}$$

Now $\bar{L} = \frac{1}{2} m ||\dot{\bar{\boldsymbol{x}}} + \boldsymbol{v}||^2$, is certainly an "odd" and "unusual" Lagrangian to choose for a free non-relativistic particle in the moving frame, but it does the job. One certainly has the correct equations of motion $\ddot{\bar{\boldsymbol{x}}} = 0$, and for *this* definition of energy and momentum, albeit "odd" and "unusual", the energy-momentum transformation laws are explicitly *linear*:

$$\bar{\boldsymbol{p}} = \boldsymbol{p}; \qquad \bar{E} = E - \boldsymbol{p} \cdot \boldsymbol{v}. \tag{B.108}$$

Note that we have made the quantities $\{\epsilon, \boldsymbol{\pi}\}$ and $\{\bar{\epsilon}, \bar{\boldsymbol{\pi}}\}$ simple, in fact zero, at the price of making the moving-frame Lagrangian complicated. (For some comments in a similar vein, see section II.A of reference [109].)

B.5.5 Summary

When looking at how this general framework and formalism applies to the Lorentz group we saw that there were good choices for ϵ and $\boldsymbol{\pi}$, and $\bar{\epsilon}$ and $\bar{\boldsymbol{\pi}}$, and also "bad" (or rather, sub-optimal) choices. There seems to be considerable freedom in how one picks ϵ and $\boldsymbol{\pi}$, and $\bar{\epsilon}$ and $\bar{\boldsymbol{\pi}}$, and so considerable freedom in choosing *affine* versus *linear* transformations for the 4-momentum. Can this freedom be used to improve things? If one is working in a region of parameter space that is "close" to special relativity (a "perturbative" deviation from special relativity) then linear transformations for the 4-momentum would seem to be the most appropriate choice. If one is working in a region of parameter space that is "close" to Galillean relativity (a "perturbative" deviation from Newtonian mechanics) then affine transformations for the 4-momentum would seem to be the most appropriate choice. The general situation is considerably murkier.

B.6 On-shell energy-momentum relations

In any particular inertial frame if one measures the energy E and momentum $\boldsymbol{p}$ of an on-shell particle then there will be *some* relation between them; an on-shell energy-momentum relation $E = E(\boldsymbol{p})$. One normally expects a very tight connection between the functional form of these energy-momentum relations and the functional form of the transformations between inertial frames — unfortunately this very tight connection is intimately related with adopting the relativity principle, and will in general fail once the relativity principle is abandoned. That is, in Lorentz violating theories the functional form of the energy-momentum relations and the functional form of the transformations between inertial frames can be (and often are) independent of each other.

B.6.1 Rest energy without the relativity principle

To see how this comes about, consider the preferred (aether) frame F, and in that frame suppose you measure the energy E and momentum $\boldsymbol{p}$ of the same particle in a number of different kinematic states to map out the energy-momentum relation $E = E(\boldsymbol{p})$ in the aether frame. To each momentum $\boldsymbol{p}$ we associate a 3-velocity $\boldsymbol{v} = \partial E/\partial \boldsymbol{p}$. Now go to the rest frame $\bar{F}$ of the particle (of course the rest frame of the particle is moving with respect to the aether). In the rest frame the particle will by definition have 3-velocity zero

$\bar{\boldsymbol{v}} = 0$, and will have some energy, call it the rest-energy $\bar{E} = E_0$ and some momentum, call it the rest-momentum $\bar{\boldsymbol{p}} = \boldsymbol{p}_0$.

If the relativity principle holds then the rest-energy and rest-momentum must be intrinsic properties of the particle that cannot depend on its velocity with respect to the aether — and in particular the rest-momentum is most typically chosen to be zero. But once one has preferred frame effects the rest-energy and rest-momentum can very definitely depend on the state of motion with respect to the aether. That is, generally we will have $\bar{E} = E_0(\bar{F})$ and $\bar{\boldsymbol{p}} = \boldsymbol{p}_0(\bar{F})$.

Transforming back to the aether frame we now see

$$E = \gamma E_0(\bar{F}) + \boldsymbol{p}_0(\bar{F})^T\, \Sigma \boldsymbol{v} + \gamma\bar{\epsilon} + \bar{\boldsymbol{\pi}}^T\, \Sigma \boldsymbol{v} - \epsilon, \tag{B.109}$$

and

$$\boldsymbol{p} = \gamma E_0(\bar{F})\boldsymbol{u} + \Sigma^T\, \boldsymbol{p}_0(\bar{F}) + \gamma\bar{\epsilon}\boldsymbol{u} + \Sigma^T \bar{\boldsymbol{\pi}} - \boldsymbol{\pi}. \tag{B.110}$$

In general, unless further assumptions are made, this is the best one can do.

We now use the freedom to choose the quantities $\{\epsilon, \boldsymbol{\pi}\}$ and $\{\bar{\epsilon}, \bar{\boldsymbol{\pi}}\}$ to make life as simple as possible. Without any real loss of generality we can choose $\bar{\boldsymbol{\pi}} = -\boldsymbol{p}_0(\bar{F})$ in which case

$$E = \gamma E_0(\bar{F}) + \gamma\bar{\epsilon} - \epsilon, \tag{B.111}$$

and

$$\boldsymbol{p} = \gamma E_0(\bar{F})\boldsymbol{u} + \gamma\bar{\epsilon}\boldsymbol{u} - \boldsymbol{\pi}. \tag{B.112}$$

(This is equivalent to choosing conventions so that in the rest frame the total "effective" rest momentum $\boldsymbol{p}_0 + \bar{\boldsymbol{\pi}} = \boldsymbol{0}$.) Let us now for definiteness choose $\epsilon = 0$ and $\boldsymbol{\pi} = \boldsymbol{0}$, then

$$E = \gamma[E_0(\bar{F}) + \bar{\epsilon}], \tag{B.113}$$

and

$$\boldsymbol{p} = \gamma[E_0(\bar{F}) + \bar{\epsilon}]\boldsymbol{u} = E\boldsymbol{u}. \tag{B.114}$$

(We have done things in this manner so that it becomes clear just how general the relation $\boldsymbol{p} = E\boldsymbol{u}$ really is.) Finally choose $\bar{\epsilon} = 0$, then with these choices we can write

$$E = \gamma E_0(\bar{F}), \tag{B.115}$$

and

$$\boldsymbol{p} = \gamma E_0(\bar{F})\boldsymbol{u} = E\boldsymbol{u}. \tag{B.116}$$

Introduce an arbitrary but fixed constant c with the dimensions of velocity (not necessarily the speed of light), and some arbitrary function $\varpi(\bar{F})$. Then in the aether frame we can write

$$E^2 - \varpi||\boldsymbol{p}||^2 c^2 = \gamma^2(1 - \varpi c^2||\boldsymbol{u}||^2)E_0^2(\bar{F}). \tag{B.117}$$

In the case of exact Lorentz invariance we have $\varpi \to 1$, with the constant c being interpreted as the speed of light, and $\gamma \to 1/\sqrt{1-v^2/c^2}$, while $\boldsymbol{u} = \boldsymbol{v}/c^2$. Furthermore E_0 is then independent of $\boldsymbol{v}$, so in this case one recovers the usual kinematic relation $E^2 - ||\boldsymbol{p}||^2c^2 = E_0^2$, while (as expected) $E = \gamma E_0$ and $\boldsymbol{p} = \gamma E_0 \boldsymbol{v}/c^2 = E\boldsymbol{v}/c^2$. In the absence of Lorentz invariance one generically has to live with the more complicated kinematics presented above. The notion of rest energy E_0 still makes perfectly good sense, but the rest energy can depend on the particle's state of motion through the aether, $E_0(\bar{F})$, and the relation to 4-momentum is considerably more subtle than one might have expected.

The key point here is that the energy-momentum relation $E(\boldsymbol{p})$ and the transformation matrix M are in general independent of each other; knowing one does not necessarily give you the other (except when Lorentz invariance is assumed, or some similar restriction is imposed). There are two additional special cases of considerable interest, which we now discuss.

B.6.2 Invariant rest energy without the relativity principle

One can speculate or hypothesize that for unknown reasons the internal structure of elementary and composite particles self-regulates so that rest energies are independent of one's state of motion through the aether. One still has rather unusual behaviour in that

$$E = \gamma E_0; \qquad \boldsymbol{p} = \gamma E_0 \boldsymbol{u} = E\boldsymbol{u}; \tag{B.118}$$

while

$$E^2 - \varpi ||\boldsymbol{p}||^2 c^2 = \gamma^2 (1 - \varpi c^2 ||\boldsymbol{u}||^2) E_0^2. \tag{B.119}$$

(Remember $\boldsymbol{u}$ is not necessarily parallel to $\boldsymbol{v}$, neither does $||\boldsymbol{u}||$ equal $||\boldsymbol{v}||$, they do not even have the same dimensions. In addition, all three of the functions $\varpi(\bar{F})$, $\gamma(\bar{F})$, and $\boldsymbol{u}(\bar{F})$ can depend on the particle's state of motion with respect to the aether. In fact ϖ is entirely arbitrary and can be adjusted to taste — we will have cause to use this freedom below.) So even with an invariant rest mass (and this is a rather strong assumption) the 4-momentum of a moving particle is rather definitely non-trivial.

B.6.3 First minimalist Lorentz-violating model

Another important special case to consider is to assume that the transformations between inertial frames are the usual Lorentz transformations but the energy-momentum relation for at least some of the particles is not Lorentz invariant. This is less bizarre than one might at first glance suspect, and is in fact the option that is in many ways most relevant to the OPERA–MINOS observations. The point is that the physical clocks and rulers we use in our laboratories have internal structures that are for all practical purposes independent of neutrino physics — and we have good phenomenological/observational evidence that (apart possibly from the neutrino sector) Lorentz invariance is an extremely good approximation to empirical reality. So it makes good sense to work in an approximation where all physical clocks and rulers are exactly Lorentz invariant, and the only Lorentz violations are hiding in the neutrino sector.[1] In this situation the rest energy of the neutrino can still depend on its state of motion with respect to the aether. In the aether frame we then have

$$E = \gamma\, E_0(\bar{F}); \qquad \boldsymbol{p} = \gamma\, E_0(\bar{F})\, \boldsymbol{v}/c^2 = E\boldsymbol{v}/c^2; \tag{B.120}$$

while

$$E^2 - ||\boldsymbol{p}||^2 c^2 = E_0^2(\bar{F}). \tag{B.121}$$

Again, even in this simplified situation, the 4-momentum and the kinematic relation are rather definitely non-trivial. If we now transform to a third inertial frame $\bar{\bar{F}}$, then certainly

$$\bar{\bar{E}}^2 - ||\bar{\bar{\boldsymbol{p}}}||^2 c^2 = E^2 - ||\boldsymbol{p}||^2 c^2 = E_0^2(\bar{F}) \tag{B.122}$$

is a Lorentz invariant, but the specific value of this Lorentz invariant quantity depends on the absolute state of motion of the neutrino as viewed from the aether frame. The way we have currently set things up, this rest energy could even be direction dependent — no isotropy assumption (at least in the neutrino sector) has yet been made. If we now add the additional assumption that neutrino physics is isotropic in the aether frame then

$$E = \gamma E_0(v); \qquad \boldsymbol{p} = \gamma E_0(v)\, \boldsymbol{v}/c^2 = E\boldsymbol{v}/c^2; \tag{B.123}$$

while

$$E^2 - ||\boldsymbol{p}||^2 c^2 = E_0^2(v). \tag{B.124}$$

So we rather explicitly see the manner in which absolute speed with respect to the aether would formally affect on-shell particle energy-momentum relations. We emphasise that in this model, even though neutrinos do not have a Lorentz invariant energy-momentum relation, their energies and 3-momenta nevertheless transform in the usual manner under Lorentz transformations. To make this look more relativistic, one could introduce a 4-velocity V_{aether} for the aether, and another 4-velocity V_ν for the neutrino. The speed v of the neutrino with respect to the aether is then the usual explicit function of the 4-inner-product $\eta(V_{\text{aether}}, V_\nu)$ and the kinematic relation takes the form

$$E^2 - ||\boldsymbol{p}||^2 c^2 = E_0^2(\eta(V_{\text{aether}}, V_\nu)). \tag{B.125}$$

This model is the first of the "minimalist" models of Lorentz violation we refer to in the abstract and introduction. It is particularly useful in that it gives one a very clean specific "target" to begin thinking about when analyzing the OPERA–MINOS observations.

[1] In contrast, if there are significant Lorentz violations in the physics underpinning one's clocks and rulers, then using the Lorentz transformations to inter-relate the space and time coordinates determined by those clocks and rulers would be a very *bad* and physically unjustified approximation.

B.6.4 Summary

We emphasise that we have gone to all this trouble in setting up a very general formalism in order to have a coherent and consistent framework to operate in once we begin to entertain possible departures from Lorentz invariance. Many of the results derived so far are quite unexpected when one has been trained to always think in a Lorentz invariant and relativity principle respecting manner.

B.7 Adding more constraints

B.7.1 Linearity plus isotropy

Now let us add the assumption of isotropy — specifically that physics is isotropic in the preferred frame, the aether frame. In particular this means that in the inertial transformation matrices

$$M = \left[\begin{array}{c|c} \gamma & -\gamma \boldsymbol{u}^T \\ \hline -\Sigma \boldsymbol{v} & \Sigma \end{array}\right]; \qquad M^{-1} = \left[\begin{array}{c|c} \gamma^{-1}(1-\boldsymbol{u}\cdot\boldsymbol{v})^{-1} & \boldsymbol{u}^T(I-\boldsymbol{v}\otimes\boldsymbol{u})^{-1}\Sigma^{-1} \\ \hline \gamma^{-1}(1-\boldsymbol{u}\cdot\boldsymbol{v})^{-1}\boldsymbol{v} & (I-\boldsymbol{v}\otimes\boldsymbol{u})^{-1}\Sigma^{-1} \end{array}\right]; \tag{B.126}$$

all vectors and matrices should be constructible only using the vector $\boldsymbol{v}$ and its magnitude — there are now assumed to be no preferred principal axes for the universe. We are also assuming that the frames F and $\bar{F}$ are "aligned" (not rotated with respect to each other). Then isotropy amounts to

$$\boldsymbol{u}||\boldsymbol{v}; \qquad \Sigma = aI + b\boldsymbol{v}\otimes\boldsymbol{v}. \tag{B.127}$$

In fact it is now useful to introduce an arbitrary but fixed unspecified constant c with the dimensions of velocity, and a dimensionless parameter ζ, to write

$$\boldsymbol{u} = \zeta \boldsymbol{v}/c^2. \tag{B.128}$$

Similarly, let us introduce dimensionless variables χ and ξ to write

$$\Sigma = \gamma\chi\boldsymbol{n}\otimes\boldsymbol{n} + \xi[I - \boldsymbol{n}\otimes\boldsymbol{n}]. \tag{B.129}$$

Recall $\boldsymbol{v} = v\boldsymbol{n}$. By appealing to isotropy, the four quantities γ, χ, ζ, and ξ are arbitrary dimensionless functions of the dimensionless variable v^2/c^2. By combining linearity with isotropy in this manner we have obtained a variant of the Robertson–Mansouri–Sexl framework; see [110, 111], and section 3.2 of [81]. (The RMS formalism invokes several other technical assumptions not relevant to the current discussion, and is not quite identical to our own framework.) Note that the quantities γ, χ, ζ, and ξ can still depend on the internal structure of one's clocks and rulers.

We now have

$$M = \left[\begin{array}{c|c} \gamma & -\gamma\zeta\boldsymbol{v}^T/c^2 \\ \hline -\gamma\chi\boldsymbol{v} & \gamma\chi\boldsymbol{n}\otimes\boldsymbol{n} + \xi[I - \boldsymbol{n}\otimes\boldsymbol{n}] \end{array}\right]. \tag{B.130}$$

An intermediate step in calculating the inverse transformation is

$$M^{-1} = \left[\begin{array}{c|c} \gamma^{-1}(1-\zeta v^2/c^2)^{-1} & \zeta\boldsymbol{v}^T(I - \zeta\boldsymbol{v}\boldsymbol{v}^T/c^2)^{-1}\Sigma^{-1}/c^2 \\ \hline \gamma^{-1}(1-\zeta v^2/c^2)^{-1}\boldsymbol{v} & (I - \zeta\boldsymbol{v}\boldsymbol{v}^T/c^2)^{-1}\Sigma^{-1} \end{array}\right]. \tag{B.131}$$

But

$$\begin{aligned} \Sigma(I - \zeta\boldsymbol{v}\boldsymbol{v}^T/c^2) &= (\gamma\chi\boldsymbol{n}\otimes\boldsymbol{n} + \xi[I - \boldsymbol{n}\otimes\boldsymbol{n}])(I - [\zeta v^2/c^2]\boldsymbol{n}\otimes\boldsymbol{n}) \\ &= \gamma\chi[1 - \zeta v^2/c^2]\boldsymbol{n}\otimes\boldsymbol{n} + \xi[I - \boldsymbol{n}\otimes\boldsymbol{n}], \end{aligned} \tag{B.132}$$

whence

$$(I - \zeta\boldsymbol{v}\boldsymbol{v}^T/c^2)^{-1}\Sigma^{-1} = \gamma^{-1}\chi^{-1}[1 - \zeta v^2/c^2]^{-1}\boldsymbol{n}\otimes\boldsymbol{n} + \xi^{-1}[I - \boldsymbol{n}\otimes\boldsymbol{n}]. \tag{B.133}$$

So the inverse transformation matrix simplifies to

$$M^{-1} = \left[\begin{array}{c|c} \gamma^{-1}(1-\zeta v^2/c^2)^{-1} & \gamma^{-1}(1-\zeta v^2/c^2)^{-1}\zeta\chi^{-1}\boldsymbol{v}^T/c^2 \\ \hline \gamma^{-1}(1-\zeta v^2/c^2)^{-1}\boldsymbol{v} & \gamma^{-1}\chi^{-1}[1-\zeta v^2/c^2]^{-1}\boldsymbol{n}\otimes\boldsymbol{n} + \xi^{-1}[I - \boldsymbol{n}\otimes\boldsymbol{n}] \end{array}\right]. \tag{B.134}$$

By a specialization of our previous discussion:

- The velocity of the moving frame with respect to the aether is $\boldsymbol{v}$.
- The velocity of the aether with respect to the moving frame is $-\chi\boldsymbol{v}$.
  (These are now at least collinear, and in fact anti-parallel, but can still differ in magnitude; they are still not equal-but-opposite.)

If we rotate to align $\boldsymbol{v}$ along the $\hat{\boldsymbol{x}}$ axis this looks a little simpler:

$$M = \left[\begin{array}{c|c|c} \gamma & -\gamma\zeta v/c^2 & \mathbf{0}^T \\ \hline -\gamma\chi v & \gamma\chi & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{0} & \xi I \end{array}\right], \tag{B.135}$$

and

$$M^{-1} = \left[\begin{array}{c|c|c} \gamma^{-1}(1-\zeta v^2/c^2)^{-1} & \gamma^{-1}(1-\zeta v^2/c^2)^{-1}\zeta\chi^{-1}v/c^2 & \mathbf{0}^T \\ \hline \gamma^{-1}(1-\zeta v^2/c^2)^{-1}v & \gamma^{-1}\chi^{-1}[1-\zeta v^2/c^2]^{-1} & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{0} & \xi^{-1}I \end{array}\right]. \tag{B.136}$$

This is as far as you can get with linearity plus isotropy — you still have four arbitrary functions $\gamma(v^2/c^2)$, $\chi(v^2/c^2)$, $\zeta(v^2/c^2)$, and $\xi(v^2/c^2)$ to deal with, but at least it is no longer an arbitrary 4×4 matrix with 16 free components. The set of transformations is still not a group, just a groupoid/pseudogroup.

In view of the isotropy assumption particle rest masses E_0 should depend only on the speed with respect to the aether, hence be of the form $E_0(v)$. Specializing our earlier discussion, with ϖ, γ and ζ being velocity dependent, in the aether frame we would have

$$E = \gamma E_0(v); \qquad \boldsymbol{p} = \gamma\zeta E_0(v)\boldsymbol{v}/c^2; \tag{B.137}$$

with

$$E^2 - \varpi||\boldsymbol{p}||^2c^2 = \gamma^2 \left[1 - \varpi\zeta^2 v^2/c^2\right] E_0(v)^2. \tag{B.138}$$

Note that Lorentz invariance corresponds to $\chi = \zeta = \xi = 1$ with $\gamma = 1/\sqrt{1 - v^2/c^2}$.

The Galilean limit is somewhat delicate: Physically we want to be looking at some sort of low velocity limit. Since when moving at zero velocity through the aether we expect $M \to I$ (corresponding to the trivial transformation) we must have $\chi(0) = \gamma(0) = \xi(0) = 1$. In contrast $\zeta(0)$ should be finite but is otherwise unconstrained. However c is at this stage just some constant with the dimensions of velocity, it does not yet have any deeper physical interpretation, so one can simply absorb $\zeta(0)$ into a redefinition of c and so effectively set $\zeta(0) \to 1$.

- In the transformation matrices M and M^{-1}, this low-velocity limit corresponds to $\zeta = \chi = \gamma = \xi = 1$, with $||\boldsymbol{v}|| \ll c$.

- Because of isotropy, in the low-velocity limit we must have both

$$\gamma(v) \approx 1 + \frac{1}{2}\gamma_2 v^2/c^2 + \dots, \quad \text{and} \quad \zeta(v) \approx 1 + \frac{1}{2}\zeta_2 v^2/c^2 + \dots. \tag{B.139}$$

  Furthermore

$$E_0(v) = E_0(0)\,\left\{1 + \frac{1}{2}\kappa_2 v^2/c^2 + \dots\right\}, \tag{B.140}$$

  so that:

$$E \approx E_0(0) + \frac{1}{2}[E_0(0)/c^2]\{\gamma_2 + \kappa_2\}v^2 + \dots; \qquad \boldsymbol{p} \approx [E_0(0)/c^2]\boldsymbol{v} + \dots \tag{B.141}$$

- If we define the low-velocity effective mass by $m_{\text{eff}} = E_0(0)/c^2$ then

$$E \approx m_{\text{eff}}c^2 + \{\gamma_2 + \kappa_2\}\,\frac{||\boldsymbol{p}||^2}{2m_{\text{eff}}} + \dots; \qquad \boldsymbol{p} \approx m_{\text{eff}}\,\boldsymbol{v} + \dots \tag{B.142}$$

So there is a sensible low-velocity limit, though it is perhaps more subtle than one might have thought.

B.7.2 Linearity plus isotropy plus reciprocity

It is sometimes useful to restrict attention to situations where $M^{-1}(\boldsymbol{v}) = M(-\boldsymbol{v})$. Note that this is an additional axiom beyond homogeneity and isotropy.

- This is (one version of) the so-called *reciprocity principle*. It is still weaker than the relativity principle.
- This version of the reciprocity principle, because it also makes assumptions about the transverse directions, is very slightly stronger than asserting that the velocity of any inertial frame as seen from the aether is minus the velocity of the aether as seen from that inertial frame [13].
- The way we have formulated it, reciprocity implies both $\chi = 1$ and $\xi = 1$, and in addition imposes the constraint

$$\gamma = \frac{1}{\sqrt{1 - \zeta v^2/c^2}}. \tag{B.143}$$

To see this, compare M with M^{-1} above, and note that $M^{-1}(\boldsymbol{v}) = M(-\boldsymbol{v})$ implies the three relations:

$$\gamma = \gamma^{-1}(1 - \zeta v^2/c^2)^{-1}; \tag{B.144}$$

$$\gamma\chi = \gamma^{-1}\chi^{-1}(1 - \zeta v^2/c^2)^{-1}; \tag{B.145}$$

$$\xi = \xi^{-1}. \tag{B.146}$$

Solving, we see

$$\xi = 1; \qquad \chi = 1; \qquad \gamma^2(1 - \zeta v^2/c^2) = 1. \tag{B.147}$$

Then

$$M = \left[\begin{array}{c|c|c} \gamma & -\gamma\zeta v/c^2 & \mathbf{0}^T \\ \hline -\gamma v & \gamma & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{0} & I \end{array}\right]; \qquad M^{-1} = \left[\begin{array}{c|c|c} \gamma & \gamma\zeta v/c^2 & \mathbf{0}^T \\ \hline \gamma v & \gamma & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{0} & I \end{array}\right]; \tag{B.148}$$

subject to the constraint

$$\gamma = \frac{1}{\sqrt{1 - \zeta v^2/c^2}}. \tag{B.149}$$

Note that you now only have one free function $\zeta(v^2/c^2)$, everything else is determined.

- Working along a somewhat different route, it has been shown [13] that combining relativity+homogeneity+isotropy implies (at least one version of) the reciprocity principle.

- Note that adopting the principle of reciprocity implies the set $\{M(\boldsymbol{v})\}$ is now closed under matrix inversion, though it is still not a group.
- This is not quite special relativity [or even Galilean relativity], but it is getting awfully close.

B.7.3 Second minimalist Lorentz-violating model

Since the model above (linearity plus isotropy plus reciprocity) is a simple one-function violation of special relativity, it holds a special place in the set of all Lorentz violating (relativity principle violating) theories — this is arguably the simplest violation of special relativity one can have at the level of the transformations between inertial frames. At the level of the coordinate transformations

$$t \to \bar{t} = \frac{t - \zeta v x/c^2}{\sqrt{1-\zeta v^2/c^2}}; \tag{B.150}$$

$$x \to \bar{x} = \frac{x - vt}{\sqrt{1-\zeta v^2/c^2}}; \tag{B.151}$$

$$y \to \bar{y} = y; \qquad z \to \bar{z} = z. \tag{B.152}$$

The closest one can get to a notion of "interval" is to observe

$$\zeta^{-1} c^2 (\Delta t)^2 - ||\Delta \boldsymbol{x}||^2 = \zeta^{-1} c^2 (\Delta \bar{t}\,)^2 - ||\Delta \bar{\boldsymbol{x}}||^2. \tag{B.153}$$

Recall that $\zeta(v)$ depends on the absolute speed of the moving frame through the aether, so this is only a 2-frame invariant, it is not a general invariant for arbitrary combinations of inertial frames. To be explicit about this, let F_1 and F_2 be two moving frames, then

$$\zeta_1^{-1} c^2 (\Delta t)^2 - ||\Delta \boldsymbol{x}||^2 = \zeta_1^{-1} c^2 (\Delta t_1\,)^2 - ||\Delta \boldsymbol{x}_1||^2, \tag{B.154}$$

and

$$\zeta_2^{-1} c^2 (\Delta t)^2 - ||\Delta \boldsymbol{x}||^2 = \zeta_2^{-1} c^2 (\Delta t_2\,)^2 - ||\Delta \boldsymbol{x}_2||^2. \tag{B.155}$$

But (ultimately due to the lack of the relativity principle, and the consequent lack of group structure for the transformations) there is, under the current assumptions, no simple relationship of this type connecting the measurements in inertial frame F_1 with those in inertial frame F_2.

When we turn to on-shell particle energy-momentum relations we still have invariant masses $E_0(v)$ that can depend on absolute velocity with respect to the aether. Therefore, in view of our previous discussion, in the aether frame we would have

$$E = \frac{E_0(v)}{\sqrt{1-\zeta v^2/c^2}}; \qquad \boldsymbol{p} = \frac{\zeta E_0(v) \boldsymbol{v}/c^2}{\sqrt{1-\zeta v^2/c^2}} = E\boldsymbol{v}/c^2; \tag{B.156}$$

with

$$E^2 - \varpi ||\boldsymbol{p}||^2 c^2 = \left[\frac{1 - \varpi\zeta^2 v^2/c^2}{1 - \zeta v^2/c^2}\right] E_0(v)^2. \tag{B.157}$$

But ϖ is a completely arbitrary function that is entirely at our disposal, so it makes sense to choose $\varpi = 1/\zeta$ in which case

$$E^2 - \zeta^{-1} ||\boldsymbol{p}||^2 c^2 = E_0(v)^2. \tag{B.158}$$

Even if we make the additional and rather stringent assumption that rest masses are invariant, independent of absolute velocity through the aether, (and this is very definitely an extra assumption beyond reciprocity), one still picks up non-trivial physics via the v-dependent function ζ:

$$E = \frac{E_0}{\sqrt{1 - \zeta v^2/c^2}}; \qquad \boldsymbol{p} = \frac{\zeta E_0 \boldsymbol{v}/c^2}{\sqrt{1 - \zeta v^2/c^2}} = E\boldsymbol{v}/c^2; \tag{B.159}$$

with

$$E^2 - \zeta^{-1} ||\boldsymbol{p}||^2 c^2 = E_0^2. \tag{B.160}$$

This model is the second of the "minimalist" models of Lorentz violation we refer to in the abstract and introduction. It is particularly useful in that it gives one a very clean specific "target" to take aim at.

B.7.4 Linearity plus isotropy plus reciprocity plus relativity

If we now (finally) adopt the relativity principle, then for arbitrary $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ the object $M(\boldsymbol{v}_2, \boldsymbol{v}_1)$ must equal $M(\boldsymbol{w})$ for *some* $\boldsymbol{w}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ (with $\boldsymbol{w}$ being interpreted as the relative velocity of the two inertial frames). But this then implies that the set $\{M(\boldsymbol{v})\}$ forms a group, not merely a groupoid/pseudogroup. We shall see that this group condition implies $\zeta = 1$, whence finally $\gamma = 1/\sqrt{1 - v^2/c^2}$. But c was some arbitrary quantity with the dimensions of velocity, it was not pre-judged to be the physical speed of light. Finite c gives you the Lorentz group, infinite c gives the Galileo group. (And the exceptional case $c^2 < 0$ actually means one is in Euclidean signature, and one obtains the $SO(4)$ rotation group. This exceptional case is normally excluded by appeal to a "pre-causality" principle [25].)

As an explicit check, assuming linearity+isotropy+reciprocity we have

$$M_1 = \left[\begin{array}{c|c|c} \gamma_1 & -\gamma_1\zeta_1 v_1/c^2 & \boldsymbol{0}^T \\ \hline -\gamma_1 v_1 & \gamma_1 & \boldsymbol{0}^T \\ \hline \boldsymbol{0} & \boldsymbol{0} & I \end{array}\right]; \qquad M_2 = \left[\begin{array}{c|c|c} \gamma_2 & -\gamma_2\zeta_2 v_2/c^2 & \boldsymbol{0}^T \\ \hline -\gamma_2 v_2 & \gamma_2 & \boldsymbol{0}^T \\ \hline \boldsymbol{0} & \boldsymbol{0} & I \end{array}\right]; \tag{B.161}$$

subject to the constraint

$$\gamma_i = \frac{1}{\sqrt{1 - \zeta_i v_i^2/c^2}}. \tag{B.162}$$

Then the group property requires the existence of some v_{12} such that

$$M_1 M_2 = M_{12}. \tag{B.163}$$

Explicitly:

$$\left[\begin{array}{c|c|c} \gamma_1\gamma_2(1+\zeta_1 v_1 v_2/c^2) & -\gamma_1\gamma_2(\zeta_1 v_1+\zeta_2 v_2)/c^2 & \mathbf{0}^T \\ \hline -\gamma_1\gamma_2(v_1+v_2) & \gamma_1\gamma_2(1+\zeta_2 v_1 v_2/c^2) & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{0} & I \end{array}\right] = \left[\begin{array}{c|c|c} \gamma_{12} & -\gamma_{12}\zeta_{12} v_{12}/c^2 & \mathbf{0}^T \\ \hline -\gamma_{12} v_{12} & \gamma_{12} & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{0} & I \end{array}\right]. \tag{B.164}$$

But by comparing the diagonal elements this can be true only if $\zeta_1 = \zeta_2$ for all values of v_1 and v_2. That is, there exists some *velocity independent constant* ζ_0 such that

$$\zeta_1 = \zeta_2 = \zeta_0. \tag{B.165}$$

This now implies

$$M_1 = \left[\begin{array}{c|c|c} \gamma_1 & -\gamma_1\zeta_0 v_1/c^2 & \mathbf{0}^T \\ \hline -\gamma_1 v_1 & \gamma_1 & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{0} & I \end{array}\right]; \qquad M_2 = \left[\begin{array}{c|c|c} \gamma_2 & -\gamma_2\zeta_0 v_2/c^2 & \mathbf{0}^T \\ \hline -\gamma_2 v_2 & \gamma_2 & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{0} & I \end{array}\right]. \tag{B.166}$$

The statement $M_1 M_2 = M_{12}$ becomes

$$\left[\begin{array}{c|c|c} \gamma_1\gamma_2(1+\zeta_0 v_1 v_2/c^2) & -\gamma_1\gamma_2\zeta_0(v_1+v_2)/c^2 & \mathbf{0}^T \\ \hline -\gamma_1\gamma_2(v_1+v_2) & \gamma_1\gamma_2(1+\zeta_0 v_1 v_2/c^2) & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{0} & I \end{array}\right] = \left[\begin{array}{c|c|c} \gamma_{12} & -\gamma_{12}\zeta_0 v_{12}/c^2 & \mathbf{0}^T \\ \hline -\gamma_{12} v_{12} & \gamma_{12} & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{0} & I \end{array}\right]. \tag{B.167}$$

But now we can simply absorb ζ_0 into a redefinition of c. After all, c is at this stage just an arbitrary but fixed constant with the dimensions of velocity. Taking $c^2 \to c^2/\zeta_0$ we have

$$M_1 = \left[\begin{array}{c|c|c} \gamma_1 & -\gamma_1 v_1/c^2 & \mathbf{0}^T \\ \hline -\gamma_1 v_1 & \gamma_1 & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{0} & I \end{array}\right]; \qquad M_2 = \left[\begin{array}{c|c|c} \gamma_2 & -\gamma_2 v_2/c^2 & \mathbf{0}^T \\ \hline -\gamma_2 v_2 & \gamma_2 & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{0} & I \end{array}\right]; \tag{B.168}$$

$$M_1 M_2 = \left[\begin{array}{c|c|c} \gamma_1\gamma_2(1+v_1 v_2/c^2) & -\gamma_1\gamma_2(v_1+v_2)/c^2 & \mathbf{0}^T \\ \hline -\gamma_1\gamma_2(v_1+v_2) & \gamma_1\gamma_2(1+v_1 v_2/c^2) & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{0} & I \end{array}\right] = \left[\begin{array}{c|c|c} \gamma_{12} & -\gamma_{12} v_{12}/c^2 & \mathbf{0}^T \\ \hline -\gamma_{12} v_{12} & \gamma_{12} & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{0} & I \end{array}\right]. \tag{B.169}$$

If c^2 is finite and positive, we have the Lorentz transformations. If c^2 is infinite we have Galileo's transformations. This is (essentially) von Ignatowsky's result. (Note that

$c^2 = 0$ is hopelessly diseased,[2] while $c^2 < 0$ actually corresponds to Euclidean signature spacetime, with the set $\{M\}$ being the group $SO(4)$ of Euclidean rotations.)

B.8 Conclusions

We have seen that once one for any reason moves away from Lorentz invariance, and specifically once one discards the relativity principle, many of the intuitions one has been trained to develop in a special relativistic setting need to be significantly and carefully revised. In a companion article we had considered threshold phenomena [103], which can be studied by picking and working in a particular arbitrary but fixed inertial frame. In the current article we have carefully analyzed what happens to the transformation properties between inertial frames once the relativity principle is abandoned. A key message to take from the above is that the situation is not hopeless — even in the absence of a relativity principle quite a lot can still be said regarding the transformation properties between inertial frames, the combination of 3-velocities, the transformation of 4-momenta, and the interplay between the energy-momentum relations for on-shell particles and the transformation properties between inertial frames.

Key features of the analysis are the groupoid/pseudo-group structure of the set of transformations, the fact that 4-momentum transforms affinely as a dual vector, the fact that there are a number of distinct stages by which Lorentz invariance can be recovered — by successively imposing linearity, then isotropy, then reciprocity, and finally the relativity principle. The net result is a coherent framework within which Lorentz symmetry breaking can be explored in a controlled and consistent manner. Overall, this article and the companion paper [103], provide general techniques of interest when analyzing the OPERA–MINOS observations.

Acknowledgments

This research was supported by the Marsden Fund, administered by the Royal Society of New Zealand. Additionally, Valentina Baccetti acknowledges support by a Victoria University PhD scholarship, and Kyle Tate acknowledges support by a Victoria University MSc scholarship.

[2] It is at this stage, setting $c^2 \to 0$, that one could if desired obtain Carroll kinematics [104, 105, 106, 107, 108] by enforcing the particular limit $c^2 \to 0$, while $v \to 0$, but holding the slowness $u = v/c^2$ fixed. The relevance to "real world" physics seems somewhat tenuous.

B.9 Some matrix identities

Herein we collect some useful matrix identities of a purely technical nature. First note that

$$(I-\boldsymbol{v}\otimes\boldsymbol{u})^{-1} = I+\sum_{n=1}^{\infty}(\boldsymbol{v}\otimes\boldsymbol{u})^n = I+(\boldsymbol{v}\otimes\boldsymbol{u})\sum_{n=1}^{\infty}(\boldsymbol{u}\cdot\boldsymbol{v})^{n-1} = I+\frac{\boldsymbol{v}\otimes\boldsymbol{u}}{1-\boldsymbol{u}\cdot\boldsymbol{v}}, \quad \text{(B.170)}$$

with this particular derivation holding for $|\boldsymbol{u}\cdot\boldsymbol{v}|<1$, though the result itself

$$(I-\boldsymbol{v}\otimes\boldsymbol{u})^{-1} = I+\frac{\boldsymbol{v}\otimes\boldsymbol{u}}{1-\boldsymbol{u}\cdot\boldsymbol{v}}, \quad \text{(B.171)}$$

holds for $\boldsymbol{u}\cdot\boldsymbol{v}\neq 1$, as can easily be verified by multiplying both sides of the equation above by $(I-\boldsymbol{v}\otimes\boldsymbol{u})$ and noting that $\det(I-\boldsymbol{v}\otimes\boldsymbol{u}) = 1-\boldsymbol{v}\cdot\boldsymbol{u}$. (The case $\boldsymbol{u}\cdot\boldsymbol{v}=1$ is the kinematic singularity alluded to previously.) Therefore

$$\boldsymbol{u}^T(I-\boldsymbol{v}\otimes\boldsymbol{u})^{-1} = \boldsymbol{u}^T+\frac{(\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{u}^T}{1-\boldsymbol{u}\cdot\boldsymbol{v}} = \frac{\boldsymbol{u}^T}{1-\boldsymbol{u}\cdot\boldsymbol{v}}, \quad \text{(B.172)}$$

at least for $\boldsymbol{u}\cdot\boldsymbol{v}\neq 1$. Similarly

$$(1-\boldsymbol{u}\cdot\boldsymbol{v})(I-\boldsymbol{v}\otimes\boldsymbol{u})^{-1} = (1-\boldsymbol{u}\cdot\boldsymbol{v})I+\boldsymbol{v}\otimes\boldsymbol{u}, \quad \text{(B.173)}$$

for $\boldsymbol{u}\cdot\boldsymbol{v}\neq 1$. Secondly observe

$$\begin{aligned}
(I-\boldsymbol{v}\otimes\boldsymbol{u})(I-\dot{\boldsymbol{x}}\otimes\boldsymbol{u})^{-1} &= (I-\boldsymbol{v}\otimes\boldsymbol{u})\left(I+\sum_{n=1}^{\infty}(\dot{\boldsymbol{x}}\otimes\boldsymbol{u})^n\right)\\
&= I-\boldsymbol{v}\otimes\boldsymbol{u}\left(\sum_{n=0}^{\infty}(\dot{\boldsymbol{x}}\cdot\boldsymbol{u})^n\right)+\dot{\boldsymbol{x}}\otimes\boldsymbol{u}\left(\sum_{n=0}^{\infty}(\dot{\boldsymbol{x}}\cdot\boldsymbol{u})^n\right)\\
&= I-\frac{\boldsymbol{v}\otimes\boldsymbol{u}}{1-\dot{\boldsymbol{x}}\cdot\boldsymbol{u}}+\frac{\dot{\boldsymbol{x}}\otimes\boldsymbol{u}}{1-\dot{\boldsymbol{x}}\cdot\boldsymbol{u}}\\
&= I+\frac{(\dot{\boldsymbol{x}}-\boldsymbol{v})\otimes\boldsymbol{u}}{1-\dot{\boldsymbol{x}}\cdot\boldsymbol{u}}, \quad \text{(B.174)}
\end{aligned}$$

with this particular derivation holding for $|\dot{\boldsymbol{x}}\cdot\boldsymbol{u}|<1$, though the result itself holds for $\dot{\boldsymbol{x}}\cdot\boldsymbol{u}\neq 1$. Therefore, for $\dot{\boldsymbol{x}}\cdot\boldsymbol{u}\neq 1$, we have

$$\begin{aligned}
(I-\boldsymbol{v}\otimes\boldsymbol{u})(I-\dot{\boldsymbol{x}}\otimes\boldsymbol{u})^{-1}(\dot{\boldsymbol{x}}-\boldsymbol{v}) &= \left(I+\frac{(\dot{\boldsymbol{x}}-\boldsymbol{v})\otimes\boldsymbol{u}}{1-\dot{\boldsymbol{x}}\cdot\boldsymbol{u}}\right)(\dot{\boldsymbol{x}}-\boldsymbol{v})\\
&= (\dot{\boldsymbol{x}}-\boldsymbol{v})+\frac{(\dot{\boldsymbol{x}}-\boldsymbol{v})\boldsymbol{u}\cdot(\dot{\boldsymbol{x}}-\boldsymbol{v})}{1-\dot{\boldsymbol{x}}\cdot\boldsymbol{u}}\\
&= (\dot{\boldsymbol{x}}-\boldsymbol{v})\left\{\frac{1-\dot{\boldsymbol{x}}\cdot\boldsymbol{u}+\boldsymbol{u}\cdot(\dot{\boldsymbol{x}}-\boldsymbol{v})}{1-\dot{\boldsymbol{x}}\cdot\boldsymbol{u}}\right\}\\
&= (\dot{\boldsymbol{x}}-\boldsymbol{v})\left\{\frac{1-\boldsymbol{u}\cdot\boldsymbol{v}}{1-\dot{\boldsymbol{x}}\cdot\boldsymbol{u}}\right\}. \quad \text{(B.175)}
\end{aligned}$$

B.10 Consistency of dynamics and kinematics

Note that from Hamilton's equations we know $\dot{\boldsymbol{x}} = \partial H/\partial \boldsymbol{p}$, so to first order (which is all we require) $\Delta E = \dot{\boldsymbol{x}} \cdot \Delta \boldsymbol{p}$. Then from our discussion of the energy-momentum transformation laws, and specifically the fact that energy-momentum differences transform linearly, we have

$$\dot{\boldsymbol{x}} \cdot \Delta \boldsymbol{p} = \gamma\, \dot{\bar{\boldsymbol{x}}} \cdot \Delta \bar{\boldsymbol{p}} + \Delta \bar{\boldsymbol{p}}^T \Sigma \boldsymbol{v} = \Delta \bar{\boldsymbol{p}} \cdot (\gamma \dot{\bar{\boldsymbol{x}}} + \Sigma \boldsymbol{v}), \tag{B.176}$$

and

$$\Delta \boldsymbol{p} = \gamma\, (\dot{\bar{\boldsymbol{x}}} \cdot \Delta \bar{\boldsymbol{p}})\boldsymbol{u} + \Sigma^T \Delta \bar{\boldsymbol{p}} = (\gamma\, \boldsymbol{u} \otimes \dot{\bar{\boldsymbol{x}}} + \Sigma^T)\Delta \bar{\boldsymbol{p}}. \tag{B.177}$$

But then, for arbitrary $\Delta \bar{\boldsymbol{p}}$

$$\left\{ \dot{\boldsymbol{x}}^T (\gamma\, \boldsymbol{u} \otimes \dot{\bar{\boldsymbol{x}}} + \Sigma^T) - (\gamma \dot{\bar{\boldsymbol{x}}}^T + \Sigma \boldsymbol{v}^T) \right\} \Delta \bar{\boldsymbol{p}} = 0, \tag{B.178}$$

implying

$$\dot{\boldsymbol{x}}^T (\gamma\, \boldsymbol{u} \otimes \dot{\bar{\boldsymbol{x}}} + \Sigma^T) = (\gamma \dot{\bar{\boldsymbol{x}}}^T + \boldsymbol{v}^T \Sigma^T). \tag{B.179}$$

That is

$$(\gamma\, \dot{\bar{\boldsymbol{x}}} \otimes \boldsymbol{u} + \Sigma)\dot{\boldsymbol{x}} = (\gamma \dot{\bar{\boldsymbol{x}}} + \Sigma \boldsymbol{v}) \tag{B.180}$$

whence

$$\dot{\boldsymbol{x}} = (\gamma\, \dot{\bar{\boldsymbol{x}}} \otimes \boldsymbol{u} + \Sigma)^{-1} (\gamma \dot{\bar{\boldsymbol{x}}} + \Sigma \boldsymbol{v}). \tag{B.181}$$

This is equivalent to the velocity transformation law we previously derived. (Note that $\dot{\bar{\boldsymbol{x}}} = \boldsymbol{0}$ implies $\dot{\boldsymbol{x}} = \boldsymbol{v}$, while $\dot{\boldsymbol{x}} = \boldsymbol{0}$ implies $\dot{\bar{\boldsymbol{x}}} = -\Sigma \boldsymbol{v}/\gamma$.)

It is perhaps easier to start from the inverse transformations

$$\Delta E \to \Delta \bar{E} = \frac{\Delta E - \Delta \boldsymbol{p} \cdot \boldsymbol{v}}{\gamma(1 - \boldsymbol{u} \cdot \boldsymbol{v})}, \tag{B.182}$$

and

$$\Delta \boldsymbol{p} \to \Delta \bar{\boldsymbol{p}} = (\Sigma^{-1})^T (I - \boldsymbol{u} \otimes \boldsymbol{v})^{-1} (\Delta \boldsymbol{p} - \Delta E \boldsymbol{u}). \tag{B.183}$$

The energy transformation equation implies

$$\dot{\bar{\boldsymbol{x}}} \cdot \Delta \bar{\boldsymbol{p}} = \frac{(\dot{\boldsymbol{x}} - \boldsymbol{v}) \cdot \Delta \boldsymbol{p}}{\gamma(1 - \boldsymbol{u} \cdot \boldsymbol{v})}, \tag{B.184}$$

while the momentum transformation equation yields

$$\begin{aligned} \Delta \bar{\boldsymbol{p}} &= (\Sigma^{-1})^T (I - \boldsymbol{u} \otimes \boldsymbol{v})^{-1} (\Delta \boldsymbol{p} - [\dot{\boldsymbol{x}} \cdot \Delta \boldsymbol{p}] \boldsymbol{u}) &\text{(B.185)} \\ &= (\Sigma^{-1})^T (I - \boldsymbol{u} \otimes \boldsymbol{v})^{-1} (I - \boldsymbol{u} \otimes \dot{\boldsymbol{x}}) \Delta \boldsymbol{p}. &\text{(B.186)} \end{aligned}$$

But then

$$\left\{ \dot{\bar{\boldsymbol{x}}}^T (\Sigma^{-1})^T (I - \boldsymbol{u} \otimes \boldsymbol{v})^{-1} (I - \boldsymbol{u} \otimes \dot{\boldsymbol{x}}) - \frac{(\dot{\boldsymbol{x}} - \boldsymbol{v})^T}{\gamma(1 - \boldsymbol{u} \cdot \boldsymbol{v})} \right\} \Delta \boldsymbol{p} = 0, \tag{B.187}$$

whence

$$\dot{\bar{\boldsymbol{x}}}^T(\Sigma^{-1})^T(I-\boldsymbol{u}\otimes\boldsymbol{v})^{-1}(I-\boldsymbol{u}\otimes\dot{\boldsymbol{x}}) = \frac{(\dot{\boldsymbol{x}}-\boldsymbol{v})^T}{\gamma(1-\boldsymbol{u}\cdot\boldsymbol{v})}. \tag{B.188}$$

Therefore

$$(I-\dot{\boldsymbol{x}}\otimes\boldsymbol{u})(I-\boldsymbol{v}\otimes\boldsymbol{u})^{-1}\Sigma^{-1}\dot{\bar{\boldsymbol{x}}} = \frac{\dot{\boldsymbol{x}}-\boldsymbol{v}}{\gamma(1-\boldsymbol{u}\cdot\boldsymbol{v})}, \tag{B.189}$$

and we see

$$\dot{\bar{\boldsymbol{x}}} = \Sigma(I-\boldsymbol{v}\otimes\boldsymbol{u})(I-\dot{\boldsymbol{x}}\otimes\boldsymbol{u})^{-1}\frac{(\dot{\boldsymbol{x}}-\boldsymbol{v})}{\gamma(1-\boldsymbol{u}\cdot\boldsymbol{v})}. \tag{B.190}$$

But (see appendix B.9)

$$(I-\boldsymbol{v}\otimes\boldsymbol{u})(I-\dot{\boldsymbol{x}}\otimes\boldsymbol{u})^{-1} \quad = \quad I+\frac{(\dot{\boldsymbol{x}}-\boldsymbol{v})\otimes\boldsymbol{u}}{1-\dot{\boldsymbol{x}}\cdot\boldsymbol{u}}. \tag{B.191}$$

Furthermore (see appendix B.9)

$$(I-\boldsymbol{v}\otimes\boldsymbol{u})(I-\dot{\boldsymbol{x}}\otimes\boldsymbol{u})^{-1}(\dot{\boldsymbol{x}}-\boldsymbol{v}) \quad = \quad (\dot{\boldsymbol{x}}-\boldsymbol{v})\left\{\frac{1-\boldsymbol{u}\cdot\boldsymbol{v}}{1-\dot{\boldsymbol{x}}\cdot\boldsymbol{u}}\right\}. \tag{B.192}$$

So finally

$$\dot{\bar{\boldsymbol{x}}} = \frac{\Sigma(\dot{\boldsymbol{x}}-\boldsymbol{v})}{\gamma(1-\boldsymbol{u}\cdot\dot{\boldsymbol{x}})}, \tag{B.193}$$

which is the 3-velocity transformation law we had previously derived. (Note that $\dot{\bar{\boldsymbol{x}}}=\mathbf{0}$ implies $\dot{\boldsymbol{x}}=\boldsymbol{v}$, while $\dot{\boldsymbol{x}}=\mathbf{0}$ implies $\dot{\bar{\boldsymbol{x}}}=-\Sigma\boldsymbol{v}/\gamma$.)

Bibliography

[1] W. A. von Ignatowsky, “Einige allgemeine Bemerkungen zum Relativitätsprinzip”, Verh. Deutsch. Phys. Ges. **12**, 788–796 (1910); B.1

[2] W. A. von Ignatowsky, “Einige allgemeine Bemerkungen zum Relativitätsprinzip”, Phys. Zeitsch. **11**, 972–976 (1910); B.1

[3] W. A. von Ignatowsky, “Das Relativitätsprinzip”, Arch. Math. Phys. **3** (17), 1–24; (18), 17–41 (1911); B.1

[4] W. A. von Ignatowsky, “Eine Bemerkung zu meiner Arbeit ‘Einige allgemeine Bemerkungen zum Relativitätsprinzip’”, Phys. Zeitsch. **12**, 779 (1911). B.1

[5] P. Frank and H. Rothe, “Ueber die Transformation der Raumzeitkoordinaten von ruhenden auf bewegte Systeme”, Ann. Phys. **34**, 825–853 (1911); B.1

[6] P. Frank and H. Rothe, “Zur Herleitung der Lorentz Transformation”, Phys. Zeitsch. **13**, 750–753 (1912). B.1

[7] A. N. Whitehead, An Enquiry Concerning the Principles of Natural Knowledge, (Cambridge, Cambridge University Press, 1919), chapter XIII. B.1

[8] L. A. Pars, “The Lorentz transformation”, Phil. Mag. **42**, 249–258 (1921). B.1

[9] V. Lalan, “Sur les postulats qui sont à la base des cinématiques”, Bull. Soc. Math. France, **65**, 83–99 (1937). B.1

[10] F. Severi, “Aspetti matematici dei legami tra relatività e senso comune”, in *Cinquant’anni di Relatività*, edited by M. Pantaleo (Firenze, Giunti, 1955), pp. 309–333. B.1

[11] Y. P. Terletskii, *Paradoxes in the Theory of Relativity* (New York, Plenum, 1968). B.1

[12] G. Süssmann, “Begründung der Lorentz-Gruppe allein mit Symmetrie- und Relativitäts-Annahmen”, Zeitsch. Naturf. **24A**, 495–498 (1969). B.1

[13] V. Berzi, V. Gorini, "Reciprocity principle and the Lorentz transformations", J. Math. Phys. **10** (1969) 1518-1524. B.1, B.7.2, B.7.2

[14] V. Gorini and A. Zecca, "Isotropy of space", J. Math. Phys. **11** (1970) 2226–2230. B.1

[15] V. Gorini, "Linear kinematical groups", Commun. Math. Phys. **21**, 150-163 (1971). B.1

[16] L. A. Lugiato, V. Gorini, "On the structure of relativity groups", J. Math. Phys. **13** (1972) 665-671. B.1

[17] V. Berzi, V. Gorini, "On space-time, reference frames and the structure of relativity groups", Annales Poincare Phys. Theor. **16** (1972) 1-22. B.1

[18] A. R. Lee and T. M. Kalotas, "Lorentz transformations from the first postulate", American Journal of Physics **43** (1975) 434–437. B.1

[19] J. M. Lévy–Leblond, "One more derivation of the Lorentz transformation", American Journal of Physics **44** (1976) 1–13. B.1

[20] W. Rindler, *Essential Relativity*, 2nd ed. (New York, Springer, 1977), pp. 51–53. B.1

[21] M. Jammer, "Some foundational problems in the special theory of relativity", in *Problems in the Foundations of Physics*, edited by G. Toraldo di Francia (Amsterdam, North-Holland, 1979), pp. 202–236. B.1

[22] R. Torretti, *Relativity and Geometry* (Dover, New York, 1996), pp. 76–82. B.1

[23] W. Rindler, *Essential Relativity, Revised Second Edition*, (Springer–Verlag, 1977). B.1, B.3.2

[24] V. Fock, *The Theory of Space, Time and Gravitation, Revised Second Edition*, (Pergamon Press, 1964). B.1, B.3.2

[25] S. Liberati, S. Sonego and M. Visser, "Faster than c signals, special relativity, and causality", Annals Phys. **298** (2002) 167 [gr-qc/0107091]. B.1, B.7.4

[26] G. Amelino-Camelia, "Relativity in space-times with short distance structure governed by an observer independent (Planckian) length scale", Int. J. Mod. Phys. D **11** (2002) 35 [gr-qc/0012051]. B.1

[27] G. Amelino-Camelia, "Particle-dependent deformations of Lorentz symmetry", arXiv:1111.5643 [hep-ph]. B.1

[28] G. Amelino-Camelia, "Doubly special relativity", Nature **418** (2002) 34 [gr-qc/0207049]. B.1

[29] S. Judes, M. Visser, "Conservation laws in 'Doubly special relativity' ", Phys. Rev. **D68** (2003) 045001. [arXiv:gr-qc/0205067 [gr-qc]]. B.1

[30] S. Liberati, S. Sonego, M. Visser, "Interpreting doubly special relativity as a modified theory of measurement", Phys. Rev. **D71** (2005) 045001. [gr-qc/0410113]. B.1

[31] D. Colladay and V. A. Kostelecky, "Lorentz-violating extension of the standard model", Phys. Rev. D **58** (1998) 116002 [arXiv:hep-ph/9809521]. B.2

[32] V. A. Kostelecky and S. Samuel, "Spontaneous Breaking of Lorentz Symmetry in String Theory", Phys. Rev. D **39** (1989) 683. B.2

[33] V. A. Kostelecky, "Gravity, Lorentz violation, and the standard model", Phys. Rev. D **69** (2004) 105009 [arXiv:hep-th/0312310]. B.2

[34] V. A. Kostelecky and R. Lehnert, "Stability, causality, and Lorentz and CPT violation", Phys. Rev. D **63** (2001) 065008 [arXiv:hep-th/0012060]. B.2

[35] V. A. Kostelecky and M. Mewes, "Signals for Lorentz violation in electrodynamics", Phys. Rev. D **66** (2002) 056005 [arXiv:hep-ph/0205211].

[36] V. A. Kostelecky and M. Mewes, "Lorentz and CPT violation in neutrinos", Phys. Rev. D **69** (2004) 016005 [arXiv:hep-ph/0309025]. B.2

[37] V. A. Kostelecky and C. D. Lane, "Constraints on Lorentz violation from clock-comparison experiments", Phys. Rev. D **60** (1999) 116010 [arXiv:hep-ph/9908504]. B.2

[38] V. A. Kostelecky and M. Mewes, "Cosmological constraints on Lorentz violation in electrodynamics", Phys. Rev. Lett. **87** (2001) 251304 [arXiv:hep-ph/0111026]. B.2

[39] D. Bear, R. E. Stoner, R. L. Walsworth, V. A. Kostelecky and C. D. Lane, "Limit on Lorentz and CPT violation of the neutron using a two-species noble-gas maser", Phys. Rev. Lett. **85** (2000) 5038 [Erratum-ibid. **89** (2002) 209902] [arXiv:physics/0007049]. B.2

[40] P. Horava, "Quantum Gravity at a Lifshitz Point", Phys. Rev. **D79** (2009) 084008. [arXiv:0901.3775 [hep-th]]. B.2

[41] M. Visser, "Lorentz symmetry breaking as a quantum field theory regulator", Phys. Rev. **D80** (2009) 025011. [arXiv:0902.0590 [hep-th]]. B.2

[42] M. Visser, "Power-counting renormalizability of generalized Horava gravity", [arXiv:0912.4757 [hep-th]]. B.2

[43] T. P. Sotiriou, M. Visser, S. Weinfurtner, "Quantum gravity without Lorentz invariance", JHEP **0910** (2009) 033. [arXiv:0905.2798 [hep-th]]. B.2

[44] T. P. Sotiriou, M. Visser, S. Weinfurtner, "Phenomenologically viable Lorentz-violating quantum gravity", Phys. Rev. Lett. **102** (2009) 251601. [arXiv:0904.4464 [hep-th]]. B.2

[45] S. Weinfurtner, T. P. Sotiriou, M. Visser, "Projectable Horava-Lifshitz gravity in a nutshell", J. Phys. Conf. Ser. **222** (2010) 012054. [arXiv:1002.0308 [gr-qc]]. B.2

[46] M. Visser, "Status of Horava gravity: A personal perspective", J. Phys. Conf. Ser. **314**, 012002 (2011). [arXiv:1103.5587 [hep-th]]. B.2

[47] C. Barceló, S. Liberati, M. Visser, "Analogue gravity", Living Rev. Rel. **8** (2005) 12. [gr-qc/0505065]. B.2

[48] M. Visser, "Acoustic black holes: Horizons, ergospheres, and Hawking radiation", Class. Quant. Grav. **15**, 1767-1791 (1998). [gr-qc/9712010]. B.2

[49] M. Visser, C. Barceló and S. Liberati, "Acoustics in Bose-Einstein condensates as an example of broken Lorentz symmetry", hep-th/0109033. B.2

[50] C. Barceló, S. Liberati and M. Visser, "Analog gravity from Bose-Einstein condensates", Class. Quant. Grav. **18** (2001) 1137 [gr-qc/0011026]. B.2

[51] D. Anselmi, "Renormalization And Lorentz Symmetry Violation", PoS C **LAQG08** (2011) 010. B.2

[52] D. Anselmi and D. Buttazzo, "Distance Between Quantum Field Theories As A Measure Of Lorentz Violation", Phys. Rev. D **84** (2011) 036012 [arXiv:1105.4209 [hep-ph]]. B.2

[53] D. Anselmi, "Renormalization of Lorentz violating theories", *Prepared for 4th Meeting on CPT and Lorentz Symmetry, Bloomington, Indiana, 8-11 Aug 2007* B.2

[54] D. Anselmi and M. Taiuti, "Vacuum Cherenkov Radiation In Quantum Electrodynamics With High-Energy Lorentz Violation", Phys. Rev. D **83** (2011) 056010 [arXiv:1101.2019 [hep-ph]]. B.2

[55] D. Anselmi and E. Ciuffoli, "Low-energy Phenomenology Of Scalarless Standard-Model Extensions With High-Energy Lorentz Violation", Phys. Rev. D **83** (2011) 056005 [arXiv:1101.2014 [hep-ph]]. B.2

[56] D. Anselmi and E. Ciuffoli, "Renormalization Of High-Energy Lorentz Violating Four Fermion Models", Phys. Rev. D **81** (2010) 085043 [arXiv:1002.2704 [hep-ph]]. B.2

[57] D. Anselmi and M. Taiuti, "Renormalization Of High-Energy Lorentz Violating QED", Phys. Rev. D **81** (2010) 085042 [arXiv:0912.0113 [hep-ph]]. B.2

[58] D. Anselmi, "Standard Model Without Elementary Scalars And High Energy Lorentz Violation", Eur. Phys. J. C **65** (2010) 523 [arXiv:0904.1849 [hep-ph]]. B.2

[59] D. Anselmi, “Weighted power counting, neutrino masses and Lorentz violating extensions of the Standard Model”, Phys. Rev. D **79** (2009) 025017 [arXiv:0808.3475 [hep-ph]]. B.2

[60] D. Anselmi, “Weighted power counting and Lorentz violating gauge theories. II: Classification”, Annals Phys. **324** (2009) 1058 [arXiv:0808.3474 [hep-th]]. B.2

[61] D. Anselmi, “Weighted power counting and Lorentz violating gauge theories. I: General properties”, Annals Phys. **324** (2009) 874 [arXiv:0808.3470 [hep-th]]. B.2

[62] D. Anselmi, “Weighted scale invariant quantum field theories”, JHEP **0802** (2008) 051 [arXiv:0801.1216 [hep-th]]. B.2

[63] D. Anselmi and M. Halat, “Renormalization of Lorentz violating theories”, Phys. Rev. D **76** (2007) 125011 [arXiv:0707.2480 [hep-th]]. B.2

[64] H. B. Nielsen and M. Ninomiya, “Beta Function In A Noncovariant Yang-mills Theory”, Nucl. Phys. B **141** (1978) 153. B.2

[65] S. Chadha and H. B. Nielsen, “Lorentz Invariance As A Low-energy Phenomenon”, Nucl. Phys. B **217** (1983) 125. B.2

[66] H. B. Nielsen and I. Picek, “Lorentz Noninvariance”, Nucl. Phys. B **211** (1983) 269 [Addendum-ibid. B **242** (1984) 542]. B.2

[67] H. B. Nielsen and I. Picek, “Lorentz Noninvariance. (addendum) On A Possible Subtraction For The Lorentz Noninvariant Model”, Nucl. Phys. B **242** (1984) 542. B.2

[68] H. B. Nielsen and I. Picek, “Redei Like Model And Testing Lorentz Invariance”, Phys. Lett. B **114** (1982) 141. B.2

[69] S. R. Coleman, S. L. Glashow, “Cosmic ray and neutrino tests of special relativity”, Phys. Lett. **B405** (1997) 249-252. [hep-ph/9703240]. B.2

B.2

[70] S. R. Coleman, S. L. Glashow, “High-energy tests of Lorentz invariance”, Phys. Rev. **D59** (1999) 116008. [hep-ph/9812418]. B.2

[71] S. Liberati, T. A. Jacobson and D. Mattingly, “High energy constraints on Lorentz symmetry violations”, arXiv:hep-ph/0110094. B.2

[72] T. Jacobson, S. Liberati and D. Mattingly, “TeV astrophysics constraints on Planck scale Lorentz violation”, Phys. Rev. D **66** (2002) 081302 [arXiv:hep-ph/0112207]. B.2

[73] T. Jacobson, S. Liberati and D. Mattingly, "Threshold effects and Planck scale Lorentz violation: Combined constraints from high energy astrophysics", Phys. Rev. D **67** (2003) 124011 [arXiv:hep-ph/0209264]. B.2

[74] D. Mattingly, T. Jacobson, S. Liberati, "Threshold configurations in the presence of Lorentz violating dispersion relations", Phys. Rev. **D67** (2003) 124012. [hep-ph/0211466]. B.2

[75] T. Jacobson, S. Liberati and D. Mattingly, "Lorentz violation and Crab synchrotron emission: A new constraint far beyond the Planck scale", Nature **424** (2003) 1019 [arXiv:astro-ph/0212190]. B.2

[76] T. Jacobson, S. Liberati and D. Mattingly, "Comments on 'Improved limit on quantum-spacetime modifications of Lorentz symmetry from observations of gamma-ray blazars' ", arXiv:gr-qc/0303001. B.2

[77] T. A. Jacobson, S. Liberati, D. Mattingly and F. W. Stecker, "New limits on Planck scale Lorentz violation in QED", Phys. Rev. Lett. **93** (2004) 021101 [arXiv:astro-ph/0309681]. B.2

[78] T. Jacobson, S. Liberati and D. Mattingly, "Quantum gravity phenomenology and Lorentz violation", Springer Proc. Phys. **98** (2005) 83 [arXiv:gr-qc/0404067]. B.2

[79] T. Jacobson, S. Liberati and D. Mattingly, "Astrophysical bounds on Planck suppressed Lorentz violation", Lect. Notes Phys. **669** (2005) 101 [arXiv:hep-ph/0407370]. B.2

[80] T. Jacobson, S. Liberati and D. Mattingly, "Lorentz violation at high energy: concepts, phenomena and astrophysical constraints", Annals Phys. **321** (2006) 150 [arXiv:astro-ph/0505267]. B.2

[81] D. Mattingly, "Modern tests of Lorentz invariance", Living Rev. Rel. **8** (2005) 5. [gr-qc/0502097]. B.2, B.7.1

[82] T. Adam *et al.* [OPERA Collaboration], "Measurement of the neutrino velocity with the OPERA detector in the CNGS beam", [arXiv:1109.4897 [hep-ex]]. B.2

[83] P. Adamson *et al.* [MINOS Collaboration], "Measurement of neutrino velocity with the MINOS detectors and NuMI neutrino beam", Phys. Rev. **D76** (2007) 072005. [arXiv:0706.0437 [hep-ex]]. B.2

[84] G. Amelino-Camelia, G. Gubitosi, N. Loret, F. Mercati, G. Rosati, P. Lipari, "OPERA — Reassessing data on the energy dependence of the speed of neutrinos", [arXiv:1109.5172 [hep-ph]]. B.2

[85] G. F. Giudice, S. Sibiryakov, A. Strumia, "Interpreting OPERA results on superluminal neutrino", [arXiv:1109.5682 [hep-ph]]. B.2

[86] A. G. Cohen and S. L. Glashow, "Pair Creation Constrains Superluminal Neutrino Propagation", Phys. Rev. Lett. **107** (2011) 181803 [arXiv:1109.6562 [hep-ph]]. B.2

[87] G. Dvali, A. Vikman, "Price for Environmental Neutrino-Superluminality", [arXiv:1109.5685 [hep-ph]]. B.2

[88] J. Alexandre, J. Ellis, N. E. Mavromatos, "On the Possibility of Superluminal Neutrino Propagation", [arXiv:1109.6296 [hep-ph]]. B.2

[89] G. Cacciapaglia, A. Deandrea, L. Panizzi, "Superluminal neutrinos in long baseline experiments and SN1987a", [arXiv:1109.4980 [hep-ph]]. B.2

[90] X. -J. Bi, P. -F. Yin, Z. -H. Yu, Q. Yuan, "Constraints and tests of the OPERA superluminal neutrinos", [arXiv:1109.6667 [hep-ph]]. B.2

[91] F. R. Klinkhamer, "Superluminal muon-neutrino velocity from a Fermi-point-splitting model of Lorentz violation", [arXiv:1109.5671 [hep-ph]]. B.2

[92] S. S. Gubser, "Superluminal neutrinos and extra dimensions: Constraints from the null energy condition", Phys. Lett. **B705** (2011) 279-281. [arXiv:1109.5687 [hep-th]]. B.2

[93] A. Kehagias, "Relativistic Superluminal Neutrinos", [arXiv:1109.6312 [hep-ph]]. B.2

[94] P. Wang, H. Wu, H. Yang, "Superluminal neutrinos and domain walls", [arXiv:1109.6930 [hep-ph]]. B.2

[95] E. N. Saridakis, "Superluminal neutrinos in Horava-Lifshitz gravity", [arXiv:1110.0697 [gr-qc]]. B.2

[96] W. Winter, "How large is the fraction of superluminal neutrinos at OPERA?", [arXiv:1110.0424 [hep-ph]]. B.2

[97] J. Alexandre, "Lifshitz-type Quantum Field Theories in Particle Physics", Int. J. Mod. Phys. **A26** (2011) 4523. [arXiv:1109.5629 [hep-ph]]. B.2

[98] F. R. Klinkhamer, G. E. Volovik, "Superluminal neutrino and spontaneous breaking of Lorentz invariance", Pisma Zh. Eksp. Teor. Fiz. **94** (2011) 731. [arXiv:1109.6624 [hep-ph]]. B.2

[99] R. Cowsik, S. Nussinov, U. Sarkar, "Superluminal Neutrinos at OPERA Confront Pion Decay Kinematics", [arXiv:1110.0241 [hep-ph]]. B.2

[100] L. Maccione, S. Liberati, D. M. Mattingly, "Violations of Lorentz invariance in the neutrino sector after OPERA", [arXiv:1110.0783 [hep-ph]]. B.2

[101] N. D. H. Dass, "OPERA, SN1987a and energy dependence of superluminal neutrino velocity", [arXiv:1110.0351 [hep-ph]]. B.2

[102] J. M. Carmona and J. L. Cortes, "Constraints from Neutrino Decay on Superluminal Velocities", arXiv:1110.0430 [hep-ph]. B.2

[103] V. Baccetti, K. Tate and M. Visser, "Lorentz violating kinematics: Threshold theorems", arXiv:1111.6340 [hep-ph]. B.2, B.8

[104] J. M. Lévy-Leblond, "Une nouvelle limite non-relativiste du groupe de Poincaré", Annales de l'Institut Henri Poincaré **A3** (1965) 1–12. B.3.3, 2

[105] Freeman Dyson, "Missed Opportunities", Bulletin of the American Mathematical Society **78** (1972) 635–652. B.3.3, 2

[106] N. A. Gromov and V. V. Kuratov, "Quantum kinematics," hep-th/0410086. B.3.3, 2

[107] Jens Madsen Houlrik and Germain Rousseaux, "'Nonrelativistic' kinematics: Particles or waves?", arXiv: 1005.1762v1 [gen-ph]. B.3.3, 2

[108] M. de Montigny and G. Rousseaux, "On some applications of Galilean electrodynamics of moving bodies", American Journal of Physics **75** (2007) 984–992. B.3.3, 2

[109] H. Padmanabhan and T. Padmanabhan, "Non-relativistic limit of quantum field theory in inertial and non-inertial frames and the Principle of Equivalence," Phys. Rev. D **84** (2011) 085018 [arXiv:1110.1314 [gr-qc]]. B.5.4

[110] H. Robertson, "Postulate versus observation in the special theory of relativity", Rev. Mod. Phys. **21** (1949) 378 B.7.1

[111] R. Mansouri, and R. U. Sexl, "A test theory of special relativity. I - Simultaneity and clock synchronization", Gen. Relativ. Gravit., 8, 497-513, (1977). B.7.1