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MATH 466/467	Applied Mathematics	Autumn 2016

Assignment

Set: February 2016 Due: May 2016

Honours-level Special Relativity

As the assignment/project for the Special Relativity module of the Honours-level Applied Mathematics paper I want you to write up *some* of the exercises in the Special Relativity for Honours notes. Note that some of the exercises are rather trivial, being designed to test your knowledge of the basics, while others are deliberately more challenging.

1. [Trivial]
Let
$$\eta \equiv \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}$$
Verify that
$$\eta^{-1} = \eta,$$
in the sense of η^{-1} being the inverse matrix for η.

2. [Easy]
Pick a specific fixed point X_1 and look at the set
$$C(X_1) = \left\{ X_2 \;\middle|\; \eta(\Delta X, \Delta X) = 0 \right\}.$$
then $C(X_1)$ is called the light-cone. (See figure overleaf.)

   Verify that $C(X_1)$ as defined above really is (both geometrically and topologically) a double cone with apex at X_1.

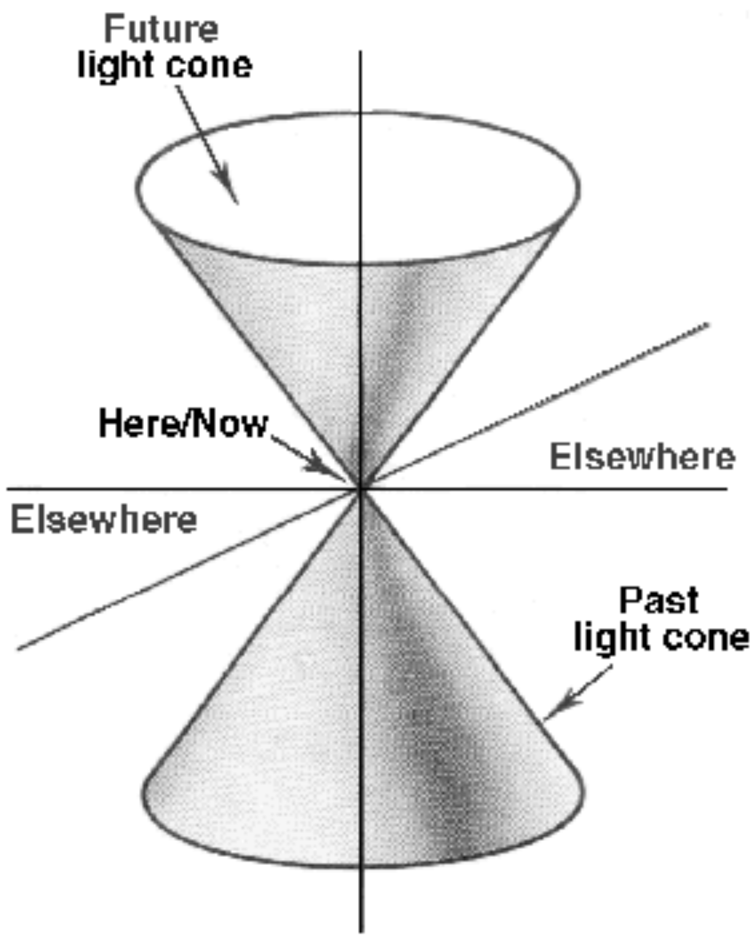

Figure 1: Light cone in special relativity. Time runs vertically up the page.

3. [Straightforward]
   Prove that $L^T \eta L \propto \eta$ is the *only* solution to the following implication:

$$\forall \Delta X \qquad (\Delta X)^T \eta (\Delta X) = 0 \qquad \Longrightarrow \qquad (\Delta X)^T [L^T \eta L] (\Delta X) = 0$$

   Hint: Take the matrix multiplications above and write them out by hand, writing $\Delta X = (c\Delta t; \Delta \vec{x})$, and so splitting spacetime into space+time.

   What happens if I reverse Δt?

   What happens if I rotate Δx?

4. [Easy]
   Verify that the set of Lorentz transformations $\mathcal{L} = \{L\}$ defined by the condition

$$\mathcal{L} = \{L \mid L^T \eta L = \eta\},$$

   forms a group under matrix multiplication.

   This is easy; at worst you will have to spend a few seconds reminding yourself of the group axioms.

5. [Straightforward]
   Explicitly verify that for any 4-vector $N = (0, \mathbf{n})$, where $\mathbf{n}$ is any 3-space unit vector satisfying $||\mathbf{n}|| = 1$, the matrix

$$L = I_4 - 2N \otimes N$$

   is a Lorentz transformation in the most general sense.

   How would you physically interpret these transformations?

6. [Straightforward]
   Explicitly verify that the three matrices

$$\begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}; \qquad \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}; \qquad \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix};$$

   are all Lorentz transformations in the most general sense.

   How would you physically interpret these transformations?

7. [Easy]
Simplify life by working in 1+1 dimensions (one space dimension, one time dimension). Take

$$\eta_2 \equiv \begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix}$$

Verify that

$$L_2 = \frac{1}{\sqrt{1-\beta^2}} \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix}$$

is the only solution to

$$L_2^T \, \eta_2 \, L_2 = \eta_2.$$

8. [Very easy]
Verify that $L_2(-\beta) = L_2(\beta)^{-1}$.

Hint: This is very easy, stop and think. If you are actually trying to invert a matrix you are doing too much work.

9. [Very easy]
Explicitly verify that for any arbitrary Lorentz transformation defined by the condition $L\,\eta\,L^T = \eta$ we have

$$\det[L] = \pm 1.$$

10. [Very easy]
Explicitly verify that for any arbitrary Lorentz transformation defined by the condition $L\,\eta\,L^T = \eta$ we have

$$L^{-1} = \eta \; L^T \; \eta.$$

11. [Straightforward; somewhat long]
Consider the Lorentz transformations in 3+1 dimensions.

(3 space dimensions, 1 time dimension.)

Adopting units where $c = 1$, restricting to motion in the x direction, and assuming no time reversal or parity flips, we have:

$$t \to t' = \gamma(t - \beta x);$$

$$x \to x' = \gamma(x - \beta t);$$
$$y \to y' = y;$$
$$z \to z' = z;$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

We can now, using column vectors, write the Lorentz transformations in matrix form as

$$X \to X' = L(\beta)\, X,$$

where

$$X = \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}; \qquad X' = \begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix}; \qquad L(\beta) = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now:

(a) Evaluate $\det[L(\beta)]$.

(b) Evaluate $[L(\beta)]^{-1}$, and interpret it.
In particular, show that $[L(\beta)]^{-1}$ is also a Lorentz transformation for *some* velocity. Which velocity?

(c) Consider an object that is at rest in the original t, x, y, z coordinates, so that its world-line may be parameterized as

$$X(t) = [t, 0, 0, 0]^T.$$

If we look at this same object using the primed coordinate system (t', x', y', z') what is its world-line

$$X'(t) = \;???$$

in this new coordinate system? From this, deduce the velocity of the object in the primed coordinate system.

(d) Now consider a pair of Lorentz transformations acting in sequence:

$$X_0 \to X_1 = L(\beta_1)\; X_0;$$

$$X_1 \to X_2 = L(\beta_2)\; X_1;$$

so that

$$X_0 \to X_2 = L(\beta_2)\; L(\beta_1)\; X_0.$$

Show by explicit matrix multiplication that the matrix product $L(\beta_2)\,L(\beta_1)$ is itself a Lorentz transformation matrix for *some* β_{12}:

$$L(\beta_{12}) = L(\beta_2)\; L(\beta_1).$$

Explicitly evaluate β_{12} as a function of β_1 and β_2:

$$\beta_{12} = f(\beta_1, \beta_2) = \;\; ???$$

(e) Defining the matrix

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

show by direct and explicit matrix multiplication that

$$L(\beta)\; \eta\; L(\beta)^T = \eta$$

for all values of β.

(f) Hence derive a formula for $[L(\beta)]^{-1}$ in terms of $L(\beta)^T$ and η.

12. [Tricky; may require a little thinking.]
Take the full Lorentz transformation and assume $v \ll c$, that is $\beta = v/c \ll 1$ so that $\gamma = 1/\sqrt{1-\beta^2} \approx 1$.

Then the Lorentz transformations reduce to

$$t' \approx t - \frac{vx}{c^2};$$

$$x' \approx x - vt;$$

$$y' = y;$$

$$z' = z.$$

These are not quite Galileo's transformations. What is that vx/c^2 term doing there? Carefully analyze the situations under which that term can safely be neglected. (It's *not* as simple as just saying $v \ll c$.)

13. [Straightforward; a little long]
Define a variable θ (typically called the "rapidity variable") by

$$\theta = \tanh^{-1}\beta = \tanh^{-1}(v/c).$$

(a) Use this definition to calculate β, γ, and $\gamma\,\beta$ as functions of rapidity

$$\beta = \ ???$$
$$\gamma = \ ???$$
$$\gamma\,\beta = \ ???$$

(b) Hence take the $1+1$ Lorentz transformation matrix

$$L(\beta) = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix}$$

and rewrite it as a function of rapidity

$$L(\theta) = \ ???$$

(c) Now consider the effect of composing two Lorentz transformations. Because the Lorentz transformations form a group, we know that their product is another Lorentz transformation. That is, for any θ_1 and θ_2 there will exist *some* θ_{12} such that

$$L(\theta_{12}) = L(\theta_2)\ L(\theta_1)$$

Explicitly verify this fact, and explicitly calculate θ_{12} as a function of θ_1 and θ_2.

$$\theta_{12}(\theta_1, \theta_2) = \ ???$$

Hint: The answer will be very simple!

(d) Using the results determined above, deduce (with proof, which is very simple) the law of composition of (collinear) velocities for special relativity:

$$v_{12}(v_1, v_2) = \ ???$$

Comment: There is a reason you will hardly ever see the law of composition of non-collinear velocities in special relativity — generically it is just too messy.

14. [Tricky]
Perform a Lorentz transformation in some arbitrary direction $\hat{n}$ (instead of along the x axis). That is $\vec{v} = v\,\hat{n} = \beta\, c\,\hat{n}$.

Show that in this case:

$$t \to t' = \gamma\left(t - \frac{\vec{v}\cdot\vec{x}}{c^2}\right);$$

$$\vec{x} \to \vec{x}' = [I - \hat{n}\otimes\hat{n}]\,\vec{x} + \gamma(\vec{x}\cdot\hat{n} - vt)\,\hat{n}.$$

Hint: Stop, think. Look at this from the appropriate direction.

15. [Straightforward]
Define the Poincare transformations as the collection of ordered pairs $P = (L, a)$ where L is a Lorentz transformation and a is a 4-vector, to be thought of as a translation in spacetime. We are to think of the Poincare transformations as acting on spacetime position according to the rule

$$X \to P(X) = LX + a$$

Identify a natural way of defining a "multiplication/ composition" operation on Poincare transformations and verify that the Poincare transformations form a group under this operation.

16. [Straightforward]
Now verify that your physically natural definition of the "multiplication/ composition" operation on Poincare transformations not only makes the Poincare transformations a group — it also makes the Poincare transformations a *semi-group* in the sense defined in the notes.

17. [Straightforward]
Consider the two definitions:

- An event X_1 *chronologically precedes* an event X_2, denoted $X_1 < X_2$ if both

$$\eta(\Delta X, \Delta X) < 0$$

and

$$\Delta t > 0.$$

- An event X_1 *causally precedes* an event X_2, denoted $X_1 \ll X_2$ if both
$$\eta(\Delta X, \Delta X) \leq 0$$
and
$$\Delta t \geq 0.$$

(a) What is the physical difference between these two definitions?

Exactly where do the two definitions differ?

(b) Look up the technical definition of a partial ordering.

Verify that both of these definitions satisfy the axioms of a partial ordering.

(You will need to distinguish a "strict partial order" from a "non-strict partial order"; for background you can always look up the discussion of partial orders in Math464/Math 465 — used there when dealing with elementary notions of topology.)

(c) Verify that these definitions are Lorentz invariant, so that any two observers will agree on chronological precedence and causal precedence. (This last step is a little trickier.)

18. [Somewhat tricky]
Define the "pseudo metric"
$$d(X, Y) = \sqrt{|\,\eta([X - Y], [X - Y])\,|}.$$
This particular definition always produce a real positive number.

(a) [Relatively easy.]
Suppose the three events X, Y, and Z are all lie in the same space-like hyperplane. Prove that the triangle inequality is satisfied.

(b) [A little trickier.]
Suppose the three events X, Y, and Z are all pairwise spacelike separated from each other. Prove that the triangle inequality is sometimes satisfied and sometimes violated.

(c) [A little trickier.]
Suppose the three events X, Y, and Z are all pairwise timelike

separated from each other. Prove that the triangle inequality is maximally violated in the sense that

$$d(X, Y) \geq d(X, Z) + d(Z, Y)$$

19. [Straightforward]
    Show that on any spacelike hyperplane a Lorentzian metric in the full Minkowski space induces a true metric on the spacelike hyperplane.

    Hint: Using translations and Lorentz transformations bring any generic spacelike hyperplane into a simple "canonical form".

20. [Straightforward]
    Show that on any timelike hyperplane there will be pairs of points that are spacelike and lightlike separated.

    Hint: Using translations and Lorentz transformations bring any generic timelike hyperplane into a simple "canonical form".

21. [Straightforward]
    Show that on any lightlike hyperplane there will be pairs of points that are spacelike separated.

    Hint: Using translations and rotations bring any generic lightlike hyperplane into a simple "canonical form".

22. [Straightforward]
    Suppose X and Z are *any* two points in Minkowski space.

    Show that $\forall \epsilon > 0$ it is possible to pick a point Y in such a way that

    $$d(X, Y) + d(Y, Z) < \epsilon.$$

    This implies that *any* two points X and Z in Minkowski space can be connected by an arbitrarily short curve.

    Note that this typically will *not* be a straight line.

23. [Straightforward]
    For a spacelike hypersurface, define the notion of a normal vector in terms of a 4-vector that is 4-orthogonal to every displacement in the hypersurface. Show that this normal vector is timelike.

24. [Straightforward]
    For a timelike hypersurface, define an appropriate notion of normal vector and demonstrate that this normal vector is spacelike.

25. [Straightforward]
    Show that every lightlike vector is orthogonal to itself.

    (Yes, this is somewhat counter-intuitive the first time you see it.)

26. [Straightforward]
    Show that for a lightlike hypersurface the normal to the hypersurface lies in the plane of the hypersurface.

    (Yes, this is somewhat counter-intuitive the first time you see it.)

27. [Easy]
    Show that for any two 4-vectors A and B the inner product

    $$\eta(A, B) = A^T \; \eta \; B$$

    is a Lorentz invariant.

28. [Tricky]
    Suppose we have two different observers, of 4-velocities

    $$V_1^a = \gamma_1(1, v_1^i) \qquad \text{and} \qquad V_2^a = \gamma_2(1; v_2^i).$$

    Show that their relative speed (the speed of one observer as seen by the other) is

    $$v = \frac{\sqrt{(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2/c^2}}{1 - \vec{v}_1 \cdot \vec{v}_2/c^2}$$

    Hint: Calculate the invariant quantity $\eta(V_1, V_2) = \eta_{ab} V_1^a V_2^b$ in two different suitably chosen reference frames.

    Hint: The case where $\vec{v}_1$ and $\vec{v}_2$ are collinear is particularly simple, and reduces to something you can readily understand in terms of results you have seen before. Its the general situation where $\vec{v}_1$ and $\vec{v}_2$ are not collinear that makes this tricky.

29. [Easy]
By Taylor series expansion, (or by using the Binomial expansion for fractional powers), at low velocities verify that the exact SR result

$$E = m_0 c^2 \, \gamma = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}},$$

approximates to

$$E \to m_0 c^2 + \frac{1}{2} m_0 v^2 + \frac{3}{8} \frac{m_0 v^4}{c^2} + O(v^6).$$

30. [Easy]
Verify that the 3-velocity of a particle can be calculated in terms of its energy and rest mass by

$$v(E) = c \, \sqrt{1 - \frac{m_0^2 \, c^4}{E^2}}.$$

31. [Easy]
Verify that the 3-velocity of a particle can be calculated in terms of its momentum and rest mass by

$$v(p) = c \, \frac{p}{\sqrt{m_0^2 \, c^2 + p^2}}.$$

32. [Easy]
Let P_1 and P_2 both be null vectors so that both $\eta(P_1, P_1) = 0$ and $\eta(P_2, P_2) = 0$.

Either:

- Find, with proof, the necessary and sufficient conditions to have $\eta(P_1, P_2) = 0$.
- Or prove that this is impossible.

Physically interpret this result.

33. [Easy]
Let P_1 and P_2 both be timelike vectors so that (in our choice of signature) $\eta(P_1, P_1) < 0$ and $\eta(P_2, P_2) < 0$.

Either:

- Find, with proof, the necessary and sufficient conditions to have $\eta(P_1, P_2) = 0$.
- Or prove that this is impossible.

Physically interpret this result.

34. [Trivial]
Relativistic Kinematics:

   (a) If $P = (E/c, p)$ is a lightlike vector, what can we say about the speed of the particle?

   (b) If $P = (E/c, p)$ is a timelike vector, what can we say about the speed of the particle?

   (c) If $P = (E/c, p)$ is a spacelike vector, what can we say about the speed of the particle?

   Hint: Stop, think. Do not blindly calculate.

35. [Substantial but straightforward.]
Relativistic Kinematics:

   Suppose we have two particles with four-momentum P_1 and P_2:

   $$P_1 = (E_1/c, p_1); \qquad P_2 = (E_2/c, p_2).$$

   (a) Calculate:
   $$\eta(P_1, P_2) = P_1^T \, \eta \, P_2 = ???$$

   (b) Interpret this Lorentz invariant quantity in two separate ways, first by going to the rest frame of particle 1; and second by going to the rest frame of particle 2.

   (c) Prove that if P_1 is timelike and P_2 is timelike, then $P_1 + P_2$ is timelike.

   (This is the situation corresponding to the four-momenta of two slower-than-light particles being added.)

   (d) Prove that if P_1 is timelike or lightlike (null), and P_2 is timelike or lightlike (null), then $P_1 + P_2$ is timelike or lightlike (null).

   (This is the situation corresponding to the four-momenta of two physical particles, either slower than light or travelling at exactly the speed of light, being added.)

(e) If $P_1 + P_2$ is lightlike, and P_1 and P_2 are both individually physical (timelike or lightlike) then this can only happen if there is a very special relationship between P_1 and P_2. What is this relationship? Prove it. Interpret the result.

36. [Tricky. This problem involves generic 4-acceleration.]
Show that in component language

$$m_0 \frac{\mathrm{d}V^a}{\mathrm{d}\tau} = \left[\delta^a{}_b + \frac{V^a \, V_b}{c^2} \right] F^b$$

So that (mass)$\times$(4-acceleration) = (the projection of the 4-force onto the three-plane perpendicular to the 4-velocity).

37. [Slightly nontrivial. This problem involves straight-line acceleration.]
A spaceship travels at some variable speed $u(\tau)$ in some fixed direction.

Its proper acceleration (measured in its own rest frame) is $a(\tau)$, and τ is the proper time. Its initial speed is u_0.

(a) Show that

$$\frac{u(\tau) - u_0}{1 - u(\tau) u_0 / c^2} = c \tanh \left\{ \frac{\int_0^\tau a(\tau') \mathrm{d}\tau'}{c} \right\}.$$

(b) Equivalently, demonstrate that

$$u(\tau) = \frac{u_0 + c \tanh \chi}{1 + u_0 \tanh \chi / c},$$

where

$$\chi = \frac{\int_0^\tau a(\tau') \mathrm{d}\tau'}{c}.$$

(c) Finally, using $\beta = u/c$, show that this is equivalent to

$$\beta(\tau) = \frac{\beta_0 + \tanh \chi}{1 + \beta_0 \tanh \chi},$$

where

$$g(\tau) = a(\tau)/c; \qquad \chi = \int_0^\tau g(\tau') \mathrm{d}\tau'$$

38. [Slightly nontrivial. This problem involves straight-line acceleration.] Consider a particle that at proper time τ_0 is located at $(t_0, x_0, 0, 0)$, and whose 4-velocity at proper time τ_0 is $V_0 = (\cosh\chi_0, \sinh\chi_0, 0, 0)$.

    Suppose it accelerates in the x direction with proper 4-acceleration $g(\tau)$. (That is, the acceleration as measured in the particle's instantaneous rest frame is $g(\tau)$.)

    Suppose further that we have adopted units so that $c \to 1$.

    Define
$$\chi(\tau) = \int_{\tau_0}^{\tau} g(\bar{\tau})\mathrm{d}\bar{\tau}.$$

    (a) Show that the spacetime trajectory [world-line] of the particle is
$$x(\tau) = x_0 + \int_{\tau_0}^{\tau} \sinh[\chi_0 + \chi(\bar{\tau})]\mathrm{d}\bar{\tau};$$
$$t(\tau) = t_0 + \int_{\tau_0}^{\tau} \cosh[\chi_0 + \chi(\bar{\tau})]\mathrm{d}\bar{\tau}.$$
    Hint: Differentiate.
    To fully solve the problem you will need to differentiate twice.

    (b) Finally, show that this is compatible with the previous question.

39. [Slightly nontrivial. This problem involves generic acceleration.] Check that in general (*without assuming $\vec{v}$ is parallel to $\vec{a}$*)
$$\begin{aligned}
\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m_0\vec{v}}{\sqrt{1 - v^2/c^2}}\right] \\
&= \frac{m_0\vec{a}}{\sqrt{1 - v^2/c^2}} + \frac{m_0\vec{v}(\vec{v}\cdot\vec{a})/c^2}{(1 - v^2/c^2)^{3/2}} \\
&= m_0\left[\gamma I + \gamma^3\frac{\vec{v}\otimes\vec{v}}{c^2}\right]\vec{a}.
\end{aligned}$$

    Notation: $(\vec{b}\otimes\vec{c})\,\vec{d} \equiv \vec{b}(\vec{c}\cdot\vec{d})$.

    Hint: If it helps, you can first check that everything works for $\vec{v}$ parallel to $\vec{a}$, and then check the general case.

40. [Slightly nontrivial. This problem involves acceleration.]
The previous question justifies the definition of an "effective relativistic mass", which is a 3×3 matrix:

$$m_{\text{eff}} = m_0 \left[\gamma I + \gamma^3 \frac{\vec{v} \otimes \vec{v}}{c^2} \right].$$

By hook or by crook establish that

$$[m_{\text{eff}}]^{-1} = \frac{\sqrt{1 - v^2/c^2}}{m_0} \left[I - \frac{\vec{v} \otimes \vec{v}}{c^2} \right].$$

Hint: Stop, think. Do not blindly calculate. Since I have given you both m_{eff} and $[m_{\text{eff}}]^{-1}$ this is really easy.

41. [Slightly nontrivial. This problem involves acceleration.]
Using the results of the previous two questions check that

$$\vec{f} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$$

implies

$$\vec{a} = \frac{\sqrt{1 - v^2/c^2}}{m_0} \left[I - \frac{\vec{v} \otimes \vec{v}^T}{c^2} \right] \vec{f}.$$

Comment: The fact that his formula is relatively complicated is one of the reasons people tend not to work with 3-force and 3-acceleration in SR — working with 4-force and 4-acceleration is often much simpler.

42. [Straightforward.]
Verify that for a charged particle in an electro-magnetic field the 3-acceleration is

$$\vec{a} = \frac{q\sqrt{1 - v^2/c^2}}{m_0} \left[\vec{E} - \frac{(\vec{v} \cdot \vec{E})\vec{v}}{c^2} + \frac{\vec{v} \times \vec{B}}{c} \right].$$

What happens to the 3-acceleration as $|\vec{v}| \to c$?

43. [Straightforward; pure linear algebra.]
Take a big square or rectangular matrix and break it apart into smaller but still rectangular blocks; these are called "partitioned matrices".

Verify that matrix multiplication satisfies the "block multiplication" rule:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BC & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

If necessary, look up a few reference books.

This is simply a standard trick of linear algebra.

Note that the order in which the matrix multiplications is carried out is important.

44. [Substantial but straightforward.]
Lorentz transformation for Electric and Magnetic fields:

In the notes I've shown you what happens if we Lorentz transform in a general direction. Now let's simplify life.

Starting from the formulae for general $\vec{\beta}$, verify that if we Lorentz transform in the x direction, so that $\vec{\beta} = (\beta, 0, 0)$, then we have the simpler results:

$$\bar{E}^x = E^x; \qquad \bar{E}^y = \gamma(E^y - \beta B^z); \qquad \bar{E}^z = \gamma(E^z + \beta B^y);$$

$$\bar{B}^x = B^x; \qquad \bar{B}^y = \gamma(B^y + \beta E^z); \qquad \bar{B}^z = \gamma(B^z - \beta E^y).$$

45. [Straightforward.]
Lorentz transformation for Electric and Magnetic fields:

Find the Lorentz transformation properties for the two quantities $|\vec{E}|^2 - |\vec{B}|^2$ and $\vec{E} \cdot \vec{B}$.

(a) That is, evaluate

$$|\vec{E}|^2 - |\vec{B}|^2 \to |\vec{\bar{E}}|^2 - |\vec{\bar{B}}|^2 = ???$$

and

$$\vec{E} \cdot \vec{B} \to \vec{\bar{E}} \cdot \vec{\bar{B}} = ???$$

(b) You should be able to do this easily enough for a general Lorentz transformation in an arbitrary direction, but there is no real loss of generality in assuming the Lorentz transformation acts along the x axis. Why? (There is a very easy argument.)

Hint: If you are getting messy complicated final results you are doing something wrong.

46. [Straightforward.]
Lorentz transformation for Electric and Magnetic fields:

    (a) Evaluate the two quantities $F^{ab}\,F_{ab}$ and $\epsilon_{abcd}\,F^{ab}\,F^{cd}$ in terms of $\vec{E}$ and $\vec{B}$.

    (b) This should make the reason for the results of the previous exercise obvious. Why?

47. [Straightforward]
Lorentz transformation for Electric and Magnetic fields:

    The general electromagnetic field is

$$F^{ab} = \frac{1}{c}\left[\begin{array}{c|c} 0 & +\vec{E} \\ \hline -\vec{E} & *B \end{array}\right] = \frac{1}{c}\left[\begin{array}{c|ccc} 0 & +E_x & +E_y & +E_z \\ \hline -E_x & 0 & +B_z & -B_y \\ -E_y & -B_z & 0 & +B_x \\ -E_z & +B_y & -B_x & 0 \end{array}\right].$$

    (a) Show that by performing a general Lorentz transformation, (boost [change of velocity] plus rotation), you can at any specified point *always* bring it into the standard form

$$F^{ab} = \frac{1}{c}\left[\begin{array}{cc|cc} 0 & +E & 0 & 0 \\ -E & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & +B \\ 0 & 0 & -B & 0 \end{array}\right].$$

    (b) Explicitly evaluate the quantities E and B in terms of the two quantities $F^{ab}\,F_{ab}$ and $\epsilon_{abcd}\,F^{ab}\,F^{cd}$.

48. [Tricky; conceptual question]
Lorentz transformation for Electric and Magnetic fields:

Consider a situation where there is no electric field, $\vec{E} = \vec{0}$, but there is a magnetic field, $\vec{B} \neq 0$. A moving electron then experiences a force

$$\vec{F} = q\, \vec{v} \times \vec{B}/c,$$

and so does not travel in a straight line, its path is deflected.

Now do a Lorentz transformation into the rest frame of the electron. In its rest frame, the 3-velocity of the particle is by definition zero. So in this frame the magnetic force seems to have vanished? What is going on here? Please explain?

49. [Straightforward; easy.]
Define

$$h^{ab} = \eta^{ab} + V^a\, V^b,$$

or equivalently

$$h^a{}_b = \delta^a{}_b + V^a\, V_b.$$

Prove that $h^a{}_b$ really is a projection operator in the sense that

$$h^a{}_b\, h^b{}_c = h^a{}_c.$$

50. [Tricky.]
Consider a fluid with a position-dependent 4-velocity vector field V^a.

Define the 4-acceleration vector field to be $A^a = V^b\, \nabla_b V^a$.

Verify that with this definition of the 4-acceleration, the 4-acceleration is everywhere 4-orthogonal to the 4-velocity.

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