8 Connectivity

We start this chapter by discussing a way to construct a matroid from two matroids on disjoint ground sets.

Proposition 8.1. *Let M_1 and M_2 be matroids with $E(M_1) \cap E(M_2) = \emptyset$. Let $\mathcal{I}$ be*

$$\{I_1 \cup I_2 \colon I_1 \in \mathcal{I}(M_1),\ I_2 \in \mathcal{I}(M_2)\}.$$

Then $\mathcal{I}$ is the family of independent sets of a matroid on the ground set $E(M_1) \cup E(M_2)$.

Proof. Since $\emptyset \in \mathcal{I}(M_1)$ and $\emptyset \in \mathcal{I}(M_2)$, the union, $\emptyset$, is a member of $\mathcal{I}$. Therefore $\mathcal{I}$ obeys **I1**. Now suppose that $I \in \mathcal{I}$, and that J is a subset of I. Then $I = I_1 \cup I_2$, where $I_1 = I \cap E(M_1)$ and $I_2 = I \cap E(M_2)$. Since I is a member of $\mathcal{I}$, it follows that $I_1 \in \mathcal{I}(M_1)$ and $I_2 \in \mathcal{I}(M_2)$. Now $J \cap E(M_1) \subseteq I_1$, so $J \cap E(M_1) \in \mathcal{I}(M_1)$, since $\mathcal{I}(M_1)$ obeys **I2**. The same argument shows that $J \cap E(M_2) \in \mathcal{I}(M_2)$. Since $J = (J \cap E(M_1)) \cup (J \cap E(M_2))$, it follows that J is a member of $\mathcal{I}$. Therefore $\mathcal{I}$ obeys **I2**. Finally, we suppose that I and J are members of $\mathcal{I}$, and that $|J| < |I|$. Let $I_1 = I \cap E(M_1)$ and $I_2 = I \cap E(M_2)$, and let $J_1 = J \cap E(M_1)$ and $J_2 = J \cap E(M_2)$. Then I_1 and J_1 are members of $\mathcal{I}(M_1)$ while I_2 and J_2 are members of $\mathcal{I}(M_2)$. If $|J_1| \geq |I_1|$ and $|J_2| \geq |I_2|$, then

$$|J| = |J_1| + |J_2| \geq |I_1| + |I_2| = |I|$$

which contradicts our assumption. Therefore, by relabelling as necessary, we can assume that $|J_1| < |I_1|$. Hence there is an element $e \in I_1 - J_1$ such that $J_1 \cup e \in \mathcal{I}(M_1)$. This means that $e \in I - J$, and $J \cup e = (J_1 \cup e) \cup J_2$ is a member of $\mathcal{I}$. Therefore $\mathcal{I}$ obeys **I3**. $\square$

The matroid from Proposition 8.1 is called the 1-*sum* or *direct sum* of M_1 and M_2, and is written $M_1 \oplus_1 M_2$ or $M_1 \oplus M_2$.

Exercise 8.2. Characterise the bases, circuits, and rank function of the direct sum $M_1 \oplus M_2$.

Proposition 8.3. *Let M_1 and M_2 be matroids with $E(M_1) \cap E(M_2) = \emptyset$. Then*

$$r_{M_1 \oplus M_2}(E(M_1)) + r_{M_1 \oplus M_2}(E(M_2)) = r(M_1 \oplus M_2).$$

Proof. It is easy to see that a basis of $M_1 \oplus M_2$ is a union of a basis of M_1 with a basis of M_2. Therefore $r(M_1 \oplus M_2) = r(M_1) + r(M_2)$. It is obvious that $(M_1 \oplus M_2)|E(M_1) = M_1$, so $r_{M_1 \oplus M_2}(E(M_1)) = r_{M_1}(E(M_1)) = r(M_1)$. The same argument shows that $r_{M_1 \oplus M_2}(E(M_2)) = r(M_2)$. The result follows. $\square$

A converse result also holds. Recall that when we say (X, Y) is a partition of the set E, then X and Y are non-empty sets, $X \cap Y = \emptyset$, and $X \cup Y = E$.

Proposition 8.4. *Let M be a matroid, and let (X, Y) be a partition of $E(M)$ such that $r(X) + r(Y) = r(M)$. Then $M = (M|X) \oplus (M|Y)$.*

Proof. We will show that $\mathcal{I}(M) = \mathcal{I}((M|X) \oplus (M|Y))$. First note that if I is an independent set of M, then $I \cap X$ and $I \cap Y$ are independent sets of $M|X$ and $M|Y$ respectively. Therefore $I = (I \cap X) \cup (I \cap Y)$ is independent in $(M|X) \oplus (M|Y)$.

Now let I_X and I_Y be independent sets of $M|X$ and $M|Y$ respectively, so that $I = I_X \cup I_Y$ is independent in $(M|X) \oplus (M|Y)$. Assume, towards a contradiction, that I is dependent in M, so it contains a circuit C. Note that C cannot be contained in I_X or I_Y, for these are independent in $M|X$ and $M|Y$ and hence in M. Therefore $C \cap X$ and $C \cap Y$ are both non-empty. Let e be an element of $C \cap X$. Now $C - e$ is independent in M, so it is contained in a basis B of M. Note that e is not in B, for otherwise B contains C. Also, $B \cap X$ and $B \cap Y$ are independent in $M|X$ and $M|Y$, respectively. If $B \cap X$ is not a basis of $M|X$, then $|B \cap X| < r(X)$. This means that

$$|B \cap Y| = |B| - |B \cap X| > r(M) - r(X) = r(Y).$$

But this is impossible, as $B \cap Y$ is independent in $M|Y$. Therefore $B \cap X$ is maximally independent in $M|X$. This means that $B \cup e$ contains a circuit C' of $M|X$, by Proposition 1.12. Note that C' is a circuit of M and $C \neq C'$, since C' is disjoint from Y, and C is not. Now B is independent in M, and $B \cup e$ is dependent, but $B \cup e$ contains two distinct circuits, C and C'. This contradicts Proposition 1.12. Therefore I is independent in M. We have proved that the independent sets of M and $(M|X) \oplus (M|Y)$ are identical, so $M = (M|X) \oplus (M|Y)$, as required. $\square$

If (X, Y) is any partition of the ground set of a matroid M, then

$$r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y) = r(E(M)) + r(\emptyset) = r(M),$$

by the submodularity of the rank function. The last two results show that this inequality is an equality precisely in the case that M can be expressed as a direct sum of non-empty matroids. This motivates the following definitions.

Definition 8.5. Let M be a matroid. A 1-*separation* of M is a partition (X, Y) of $E(M)$ such that $r(X) + r(Y) = r(M)$.

Definition 8.6. A matroid is *connected* if it does not have a 1-separation.

Propositions 8.3 and 8.4 establish the following result.

Proposition 8.7. *A matroid is connected if and only if it cannot be expressed as the direct sum of two non-empty matroids.*

Exercise 8.8. Prove that if e is a loop or a coloop of M and $|E(M)| \geq 2$, then $(\{e\}, E(M) - e)$ is a 1-separation.

Connected components. Recall that the connected components of a graph are defined as the equivalence classes of a particular relation on the vertices (see page 17). We can develop a similar idea for matroids.

Proposition 8.9. *Let M be a matroid, and let C_1 and C_2 be distinct circuits in M such that $C_1 \cap C_2 \neq \emptyset$. If $e \in C_1$ and $f \in C_2$, then there is a circuit C of M such that $\{e, f\} \subseteq C \subseteq C_1 \cup C_2$.*

Proof. Assume the result does not hold. Amongst all circuits C_1, C_2 for which the result fails, choose C_1 and C_2 so that $|C_1 \cup C_2|$ is as small as possible. Now $e \in C_1$ and $f \in C_2$, but there is no circuit containing $\{e, f\}$ and contained in $C_1 \cup C_2$. Clearly $e \notin C_2$ and $f \notin C_1$, because otherwise C_1 or C_2 is a circuit contained in $C_1 \cup C_2$ that contains both e and f. Let x be an element in $C_1 \cap C_2$. By Proposition 5.13, there is a circuit $C_3 \subseteq (C_1 \cup C_2) - x$ such that $e \in C_3$. Now C_3 must contain an element $y \in C_2 - C_1$, for otherwise C_3 is properly contained in C_1. We have assumed that no circuit in $C_1 \cup C_2$ contains both e and f. Since $e \in C_3$, it follows that f is in $C_2 - C_3$. As $y \in C_2 \cap C_3$, we can apply Proposition 5.13 to C_2 and C_3, and deduce that there is a circuit $C_4 \subseteq (C_2 \cup C_3) - y$ such that $f \in C_4$. Now $C_4 \subseteq (C_1 \cup C_2) - y$. There is at least one element in $C_1 \cap C_4$, for otherwise C_4 is properly contained in C_2. Moreover, $e \in C_1$ and $f \in C_4$. There cannot be a circuit in $C_1 \cup C_4$ that contains both e and f, for then there would be a circuit in $C_1 \cup C_2$ that contains both e and f. But $|C_1 \cup C_4| < |C_1 \cup C_2|$, since $y \notin C_1 \cup C_4$. This contradicts our choice of C_1 and C_2, which completes the proof. $\square$

For a matroid M, let $\sim$ be the relation on $E(M)$ such that, for $e, f \in E(M)$, we have $e \sim f$ if either $e = f$, or some circuit of M contains e and f. The next result follows almost immediately from Proposition 8.9.

Corollary 8.10. *Let M be a matroid. The relation $\sim$ is an equivalence relation on $E(M)$.*

A *connected component* of M is an equivalence class under $\sim$.

Proposition 8.11. *The matroid M is connected if and only if $E(M)$ is a connected component.*

Proof. Let E be the ground set of M. Assume that E is not a connected component, so M has a connected component X such that $X \neq E$. Now every circuit of M is contained in X or $E - X$. Therefore $C \subseteq E$ is a circuit of M if and only if it is a circuit of $M|X$ or $M|(E - X)$, and it is easy to see this is true if and only if C is a circuit of $(M|X) \oplus (M|(E - X))$. Therefore $M = (M|X) \oplus (M|(E - X))$. Proposition 8.3 now shows that M has a 1-separation, so it is not connected.

For the converse, assume that M is not connected, so it can be expressed as $M = M_1 \oplus M_2$, by Proposition 8.7, where M_1 and M_2 are not empty matroids. It is easy to see that no circuit of M can contain elements from $E(M_1)$ and $E(M_2)$, so it is clear that $E(M)$ is not a component. $\square$

Exercise 8.12. Prove that the partition (X, Y) is a 1-separation of the matroid M if and only if X and Y are unions of connected components.

Connectivity and graphs. Which graphs correspond to connected graphic matroids? Isolated vertices in a graph have no effect on the corresponding graphic matroid, so we will consider only graphs without isolated vertices. If a graph G contains a loop, then $M(G)$ is connected if and only if G contains no other edge. Therefore we may as well consider loopless graphs.

Let $G = (V, E)$ be a graph, and let v be a vertex of G. Then $G - v$ is the graph

$$(V - v, \{e \in E \colon v \notin e\}).$$

In other words, $G - v$ is obtained from G by deleting the vertex v, and all the edges incident with it. A vertex v of G is a *cut-vertex* if $G - v$ has more connected components than G. A graph is 2-*connected* if it is connected, and has no cut-vertices.

Proposition 8.13. *Let G be a loopless graph with no isolated vertices. Then $M(G)$ is connected if and only if G is 2-connected.*

Proof. Assume that $M(G)$ is connected. First, suppose that G is not connected. Let H_1 and H_2 be two distinct connected components of G. Since G has no isolated vertices, there are edges e_1 and e_2 of $G[H_1]$ and $G[H_2]$ respectively. But clearly no cycle of G can contain both e_1 and e_2. Therefore no circuit of $M(G)$ contains e_1 and e_2, and Proposition 8.11 states that $M(G)$ is not connected, contrary to assumption. Henceforth, we may assume that G is connected.

Suppose that G is not 2-connected. Then G has a vertex v such that $G - v$ has at least two connected components, H_1 and H_2. Since G is connected, there are paths in G from vertices in H_1 to vertices in H_2. Any such path must go through v, so there are edges e_1 and e_2 such that e_i joins v to a vertex in H_i, for $i = 1, 2$. Let $e_i = \{v, u_i\}$, for $i = 1, 2$. As $M(G)$ is connected, there is a cycle C of G that contains e_1 and e_2, by Proposition 8.11. Now $C - v$ is a path in $G - v$, and this path connects u_1 to u_2. But u_1 and u_2 are in different connected components of $G - v$, so we have a contradiction. Therefore G is 2-connected. This completes one direction of the proof.

For the converse, assume that G is 2-connected but $M(G)$ is not connected. Then $E(G)$ is not a connected component of $M(G)$, by Proposition 8.11. Let e_1 and e_2 be edges of G that are not contained in a common cycle of G. Note that G has no loops, and let e_i be $\{u_i, v_i\}$, for $i = 1, 2$. Let P be a walk joining a vertex in $\{u_1, v_1\}$ to a vertex in $\{u_2, v_2\}$, and amongst all such walks assume that P has been chosen so that it has the smallest possible number of edges. The walk P exists since G is connected. By relabelling, we can assume that P joins u_1 to u_2. Now neither v_1 nor v_2 is contained in P, for otherwise our assumption on the length of P is contradicted. Moreover, P is a path, for the same reason.

Assume that P contains an edge e_3. If there were no cycle containing e_1 and e_3, then we would have chosen that pair instead of e_1 and e_2, since e_1 and e_3 are joined by a shorter path. Hence there is a cycle C_1 containing e_1 and e_3. Similarly, the cycle C_2 contains e_2 and e_3. Since e_3 is an edge in both C_1 and C_2, Proposition 8.9 implies that there is a circuit of $M(G)$ that contains e_1 and e_2, contrary to our assumption. Therefore P contains no edges, so it is a single vertex, u. This implies $u_1 = u_2 = u$ and $e_i = \{u, v_i\}$, for $i = 1, 2$. Note that $G - u$ is connected, since G is 2-connected. Therefore there is a path from v_1 to v_2 in $G - u$. But this path, along with the edges e_1 and e_2, makes a cycle of G that contains e_1 and e_2. This final contradiction completes the proof. □

Higher connectivity. Next we extend our notion of connectivity. Let M be a matroid. Recall that if X and Y are disjoint sets such that $X \cup Y = E(M)$, then $r(X) + r(Y) \geq r(M)$, or equivalently $r(X) + r(Y) - r(M) \geq 0$. Moreover, (X, Y) is a 1-separation if and only if $|X|, |Y| \geq 1$ and $r(X) + r(Y) - r(M) < 1$. This inspires the following extension of the definition.

Definition 8.14. Let M be a matroid with ground set E. For $X \subseteq E$, let

$$\lambda_M(X) = r(X) + r(E - X) - r(M).$$

We call $\lambda_M(X)$ the *connectivity function* of M.

Definition 8.15. Let k be a positive integer. A *k-separation* of a matroid M is a partition (X, Y) of $E(M)$ with the property that $|X| \geq k$ and $|Y| \geq k$, and $\lambda_M(X) < k$.

Definition 8.16. Let n be an integer that is at least two. A matroid is *n-connected* if there is no positive integer $k < n$ such that M has a k-separation.

Note that, under this definition, a 2-*connected* matroid is what we have previously called a *connected* matroid.

Next we explore some of the properties of the connectivity function of a matroid. One of its nice features is that it is invariant under duality. To see this, we first give an equivalent formulation of the connectivity function using the corank of a set.

Lemma 8.17. *Let M be a matroid and $X \subseteq E(M)$. Then*

$$\lambda_M(X) = r(X) + r^*(X) - |X|.$$

Proof. By Proposition 3.9, we have $r(E - X) = r^*(X) - |X| + r(M)$. So

$$\begin{aligned}\lambda_M(X) &= r(X) + r(E - X) - r(M)\\ &= r(X) + r^*(X) - |X| + r(M) - r(M)\\ &= r(X) + r^*(X) - |X|,\end{aligned}$$

as required. □

Corollary 8.18. *Let M be a matroid, and $X \subseteq E(M)$. Then*

$$\lambda_M(X) = \lambda_{M^*}(X).$$

The next result follows immediately.

Corollary 8.19. *A matroid M is n-connected if and only M^* is n-connected.*

Proposition 8.20. *Let M be an n-connected matroid with at least $2(n-1)$ elements. Then every circuit of M contains at least n elements.*

Proof. Let E be the ground set of M. Suppose M has a circuit C with at most $n-1$ elements. Let $k = |C|$. Then $|E - C| \geq k$, since $k \leq n-1$ and $|E| \geq 2(n-1)$. Moreover $r(C) = k-1$, and $r(E-C) \leq r(M)$. Therefore $\lambda_M(C) = r(C) + r(E-C) - r(M) < k$, so $(C, E-C)$ is a k-separation for $k < n$, which contradicts the fact that M is n-connected. $\square$

Most of our focus will be on 3-connected matroids. A matroid is *simple* if every circuit has at least three elements, and is *cosimple* if its dual is simple. Corollary 8.18 and Proposition 8.20 establish the following fact:

Proposition 8.21. *A 3-connected matroid with at least four elements is both simple and cosimple.*

Proposition 8.7 characterises connected matroids in terms of 1-sums. We can similarly characterise 3-connected matroids in terms of a sum operation.

Definition 8.22. Let M_1 and M_2 be two matroids satisfying $E(M_1) \cap E(M_2) = \{e\}$, where e is neither a loop nor a coloop in M_1 or M_2. Let $\mathcal{C}_1 = \mathcal{C}(M_1)$ and $\mathcal{C}_2 = \mathcal{C}(M_2)$. The 2-*sum* of M_1 and M_2, written $M_1 \oplus_2 M_2$, has $(E(M_1) \cup E(M_2)) - e$ as its ground set, and

$$\{C \in \mathcal{C}_1 \colon e \notin C\} \cup \{C \in \mathcal{C}_2 \colon e \notin C\} \cup \{(C_1 \cup C_2) - e \colon C_1 \in \mathcal{C}_1,\ C_2 \in \mathcal{C}_2,\ e \in C_1 \cap C_2\}$$

as its family of circuits.

Definition 8.23. Let k be a positive integer, and let (X, Y) be a k-separation of a matroid M. The separation (X, Y) is *exact* if $\lambda_M(X) = k-1$.

The following theorem is due to Seymour.

Theorem 8.24 (Seymour, 1980)**.** *If (X, Y) is an exact 2-separation of the matroid M, then there are matroids M_1 and M_2 with ground sets $X \cup e$ and $Y \cup e$ respectively (where $e \notin X \cup Y$) such that $M = M_1 \oplus_2 M_2$. Conversely, if M_1 and M_2 are matroids such that $|E(M_1)|, |E(M_2)| \geq 3$, and $E(M_1) \cap E(M_2) = \{e\}$, and e is neither a loop nor a coloop in M_1 or M_2, then $(E(M_1) - e, E(M_2) - e)$ is an exact 2-separation of $M_1 \oplus_2 M_2$.*

Corollary 8.25. *Suppose that M is a connected matroid. Then M is 3-connected if and only if it cannot be expressed in the form $M_1 \oplus_2 M_2$ such that M_1 and M_2 both have at least three elements.*

Chain Theorems and Splitter Theorems. Inductive tools for connectivity are very important. The next result is a canonical example of such an inductive tool; it ensures the existence of an element that can be removed while retaining the property of being connected.

Proposition 8.26. *Let M be a connected matroid. Then either $M \backslash e$ or M/e is connected, for any element $e \in E(M)$.*

Proof. First assume that e is a loop or coloop of M. Since M is connected, this means that $E(M) = \{e\}$. Thus $M \backslash e = M/e$ is the empty matroid, which is connected. So we may assume that e is not a loop or a coloop.

Assume that neither $M \backslash e$ nor M/e is connected. Then there are distinct elements $f, g \in E(M) - e$ such that f and g are not in a common circuit of M/e. Assume that f and g are contained in a connected component of $M \backslash e$. Call this component X. Since $M \backslash e$ is not connected, X is not equal to the ground set of $M \backslash e$. We let Y be $E(M \backslash e) - X$. Then both X and Y are non-empty, and $r_{M \backslash e}(X) + r_{M \backslash e}(Y) = r(M \backslash e)$.

By the definition of a component, there is a circuit C of $M \backslash e$ that contains f and g. Then C cannot be a circuit of M/e, since f and g are not contained in a common circuit of this matroid. Therefore Proposition 5.14 implies that $e \in \mathrm{cl}_M(C)$. Since C is a subset of X, it follows that e is also in $\mathrm{cl}_M(X)$. Therefore $r_M(X \cup e) = r_M(X) = r_{M \backslash e}(X)$. We also have $r_{M \backslash e}(Y) = r_M(Y)$ and $r(M) = r(M \backslash e)$, as e is not a coloop. Therefore

$$\begin{aligned}\lambda_M(X \cup e) &= r_M(X \cup e) + r_M(Y) - r(M) \\ &= r_{M \backslash e}(X) + r_{M \backslash e}(Y) - r(M \backslash e) = 0,\end{aligned}$$

so M has a 1-separation. This is a contradiction, as M is connected. Therefore f and g are not in the same connected component of $M \backslash e$.

Now, no circuit of $M \backslash e$ contains f and g, but f and g are contained in a circuit C of M, since M is connected. Since C is not a circuit of $M \backslash e$, it follows that $e \in C$. Thus $C - e$ is a circuit of M/e that contains f and g, by Proposition 5.14. This contradiction completes the proof. □

The analogue of Proposition 8.26 is not true for 3-connected matroids, as the next example shows. Let $n \geq 2$ be an integer. The *n-spoke wheel* (written W_n) is a graph on $n+1$ vertices. It is obtained from a cycle with n

vertices by adding a new vertex that is adjacent to every other vertex. We call the graphic matroid $M(W_n)$ the *rank-n wheel*. Note that the edges in the original cycle of W_n form a circuit-hyperplane of $M(W_n)$. The matroid obtained by relaxing this circuit-hyperplane (see page 54) is called the *rank-n whirl*, and is denoted W^n. Any wheel or whirl is 3-connected. However, if M is a rank-n wheel or whirl, and e is any element of M, then $M \backslash e$ contains a series pair, and M/e contains a parallel pair. If $n > 2$, then Proposition 8.21 implies that $M \backslash e$ and M/e are not 3-connected, for any element e.

Despite this example, there are still useful inductive results for 3-connected matroids. In fact, the next result (known as the Wheels and Whirls Theorem) shows that these matroids are the only ones to exhibit this behaviour.

Theorem 8.27 (Tutte, 1966)**.** *Let M be a non-empty 3-connected matroid. If M is not a wheel or a whirl, then there is an element e in $E(M)$ such that either $M \backslash e$ or M/e is 3-connected.*

A non-simple matroid M has a canonically associated simple matroid, called the *simplification* of M. We denote this simplification by $\mathrm{si}(M)$. Informally, $\mathrm{si}(M)$ is obtained by deleting all the loops from M, and then deleting all but one element from every parallel class. More formally, we note that a rank-one flat consists of a parallel class (and the set of all loops). We let the ground set of $\mathrm{si}(M)$ be the set of rank-one flats of M. If $\{F_1, \ldots, F_t\}$ is a set of rank-one flats of M, then the rank of $\{F_1, \ldots, F_t\}$ in $\mathrm{si}(M)$ is

$$r_M\left(\bigcup_{i=1}^{t} F_i\right).$$

This gives us the rank of any subset of the ground set of $\mathrm{si}(M)$, and therefore completely defines $\mathrm{si}(M)$.

The *cosimplification* (written $\mathrm{co}(M)$) of M is defined to be $(\mathrm{si}(M^*))^*$. Thus $\mathrm{co}(M)$ is obtained from M by contracting all coloops, and contracting all but one element from every series class. The next result is known as Bixby's Lemma.

Lemma 8.28 (Bixby, 1982)**.** *Let e be an element of the 3-connected matroid M. Then either $\mathrm{co}(M \backslash e)$ or $\mathrm{si}(M/e)$ is 3-connected.*

Results such as Proposition 8.26, Theorem 8.27, and Lemma 8.28 are sometimes known as chain theorems, since they let us find a chain of connected matroids, each one obtained from the previous one by deleting or contracting a single element (and possibly simplifying or cosimplifying). Typically, as we move into higher types of connectivity, it becomes more and more difficult to obtain chain theorems. This is why we often focus on 3-connected matroids: 3-connectivity is strong enough to impose useful structural constraints, but weak enough so that there are good inductive tools. One of the most important of these tools is the Splitter Theorem of Seymour. A *proper* minor of the matroid N is a minor that is not equal to N.

Definition 8.29. Let $\mathcal{M}$ be a minor-closed class of matroids. A *splitter* of $\mathcal{M}$ is a matroid $M \in \mathcal{M}$ such that no 3-connected member of $\mathcal{M}$ contains a proper minor isomorphic to M.

(Note that when we say that $\mathcal{M}$ is a class of matroids, we always mean that $\mathcal{M}$ is closed under isomorphism.) Equivalently, if N is a 3-connected member of $\mathcal{M}$, and N has a minor isomorphic to M, then N itself is isomorphic to M. Since no member of $\mathcal{M}$ that properly contains M as a minor can be 3-connected, we might say that M splits the members of $\mathcal{M}$ into non-3-connected matroids. It seems that verifying that M is a splitter for $\mathcal{M}$ could be an infinite task: we might need to examine all the 3-connected members of $\mathcal{M}$ and check that none of them properly contains an isomorphic copy of M as a minor. Seymour's theorem shows us that this is not the case. A *single-element extension* of a matroid M is a matroid N with the property that there is an element e in $E(N)$ such that $N\backslash e = M$. Dually, a *single-element coextension* of M is a matroid N with an element $e \in E(N)$ such that $N/e = M$. Now we can state the Splitter Theorem of Seymour. It can be seen as a strengthening of the Wheels and Whirls Theorem.

Theorem 8.30 (Seymour, 1980)**.** *Let $\mathcal{M}$ be a class of matroids. Let M be a 3-connected member of $\mathcal{M}$ with at least four elements, such that if M is a wheel or a whirl, then $\mathcal{M}$ contains no wheel or whirl with rank greater than M. Then M is a splitter for $\mathcal{M}$ if and only if there is no 3-connected single-element extension or coextension of M that is contained in $\mathcal{M}$.*

Therefore, to check that M is a splitter for $\mathcal{M}$, we do not need to check all the 3-connected members of $\mathcal{M}$, only the ones that are single-element extensions or coextensions of M. The Splitter Theorem can also be stated in the following way.

Theorem 8.31. *Let M and N be 3-connected matroids such that N has a proper minor isomorphic to M. Assume that N has at least four elements,* *and if N is a wheel or a whirl, then M does not have any wheel or whirl with rank greater than N as a minor. Then there is an element $e \in E(M)$ such that either $M\backslash e$ or M/e is 3-connected, and has a minor isomorphic to N.*

Thus we can use the Splitter Theorem to construct a chain of 3-connected matroids, each one produced from the previous one by deleting or contracting a single element. In this case, we also have the condition that all the matroids in the chain have a copy of N as a minor.