7 Representable matroids

We will start this chapter by discussing an uncomplicated (but usually impractical) method for deciding whether a matroid is representable over a given field. Recall that if A is a matrix over the field $\mathbb{F}$ with rows and columns labelled by X and Y respectively, then $M[I|A]$ is a matroid with $X \cup Y$ as its ground set. The bases of $M[I|A]$, other than X, are all subsets $Z \subseteq X \cup Y$ such that $|Z| = |X|$ and the submatrix of A with rows and columns labelled by $X - Z$ and $Y \cap Z$ has non-zero determinant. (We use $A[X - Z, Y \cap Z]$ to denote this submatrix.)

Suppose now that we seek to find a representation of a matroid M over a field $\mathbb{F}$, if such a representation exists. Let X be a basis of M, and let $Y = E(M) - X$. If a representation exists, then, as seen in Proposition 3.14, we can obtain a representation where X labels the columns of I (or, equivalently, the rows of A). So it suffices to determine if there is a matrix A over the field $\mathbb{F}$ with rows labelled by X and columns labelled by Y such that $M = M[I|A]$. We can learn something about this matrix by considering the fundamental circuits relative to the basis X; recall that, for $y \in Y$, there is a unique circuit contained in $X \cup y$ that we denote by $C(y, X)$. Recall also that, if $x \in X$ and $y \in Y$, then A_{xy} denotes the entry of A in the row labelled by x and the column labelled by y.

Proposition 7.1. *Let X be a basis of the matroid M, and $Y = E(M) - X$. Let A be a matrix with rows labelled by X and columns labelled by Y such that $M = M[I|A]$ over the field $\mathbb{F}$. If $x \in X$ and $y \in Y$, then $A_{xy} \neq 0$ if and only if $x \in C(y, X)$.*

Proof. The entry A_{xy} is non-zero if and only if the determinant of $A[\{x\}, \{y\}]$ is non-zero, which is true if and only if $(X - x) \cup y$ is a basis (by Proposition 3.13). But this is true if and only if $(X - x) \cup y$ contains no circuit, which is true if and only if x is in $C(y, X)$. □

Therefore we can determine which entries of A are non-zero through our knowledge of the circuits of M.

Multiplying a row or a column of A by an element of $\mathbb{F}$ is called *scaling*. We have already noted that if A' is obtained from A by scaling rows and columns with non-zero elements of $\mathbb{F}$, then $M[I|A'] = M[I|A]$. By scaling we can assume that certain non-zero entries in A are actually equal to one. Before proving this, we need to introduce a graph that is related to A.

Let $G(A)$ be the bipartite graph on vertex set $X \cup Y$, where each edge joins a vertex in X to a vertex in Y, and the vertices $x \in X$ and $y \in Y$

are adjacent in $G(A)$ if and only if $A_{xy} \neq 0$. Any edge e of this graph corresponds to a non-zero entry of A. We will denote this entry by A_e.

Proposition 7.2. *Let X be a basis of the matroid M, and let $Y = E(M) - X$. Let A be a matrix with rows labelled by X and columns labelled by Y such that $M = M[I|A]$ over the field $\mathbb{F}$. Let $\{e_1, \ldots, e_t\}$ be a set of edges of $G(A)$ such that $G(A)[\{e_1, \ldots, e_t\}]$ is a forest. Let $p_1, \ldots, p_t$ be a sequence of non-zero elements of $\mathbb{F}$. By scaling rows and columns of A with non-zero elements of $\mathbb{F}$ we can obtain a matrix A' such that $A'_{e_i} = p_i$ for each $i \in \{1, \ldots, t\}$.*

Proof. We prove this by induction on t. If $t = 1$, then we simply multiply the row containing A_{e_1} by $p_1 A_{e_1}^{-1}$. (Note that the inverse A_e^{-1} exists, since $A_e \neq 0$, for any edge e of $G(A)$.)

Now we assume that the result holds for collections of $t-1$ edges. Since $G(A)[\{e_1, \ldots, e_t\}]$ contains no cycles, it contains a vertex incident with exactly one edge. By relabelling, let us assume that this edge is e_t, and that v is the vertex incident with e_t and no other edges in $\{e_1, \ldots, e_t\}$. By induction, we can scale A with non-zero elements of $\mathbb{F}$ to produce a matrix A'' such that $A''_{e_i} = p_i$ for all $i \in \{1, \ldots, t-1\}$. Now v corresponds to a row or a column of A'', and we multiply that row or column by $p_t(A''_{e_t})^{-1}$ to produce the matrix A'. Since v is incident with no other edges in $i \in \{1, \ldots, t-1\}$, the property that $A'_{e_i} = p_i$ still holds for all $i \in \{1, \ldots, t-1\}$, as required. $\square$

Now we can assume that A has been scaled so that a set of entries corresponding to a maximal forest of $G(A)$ are equal to some specified constants. (We usually scale so that they are all equal to one.) The remaining non-zero entries of A can be filled with variables that represent non-zero elements of the field $\mathbb{F}$. If we know that a particular set is a basis of M, then this translates to a particular submatrix of A having a non-zero determinant, and this implies that some polynomial function of the variables is non-zero. Similarly, if we know that a set with size $r(M)$ is not a basis, then we deduce that a certain submatrix has zero determinant. This shows that some polynomial function of the variables is zero. By proceeding in this way, we produce a set of constraints on the variables. We may be able to find an assignment of values to the variables that satisfies all of the constraints. In this case, we have found a matrix A that represents M over $\mathbb{F}$. Alternatively, we may show that there is no assignment of values that satisfies all the constraints, in which case we have shown that M is not $\mathbb{F}$-representable. (Unfortunately, this strategy is inefficient, and quickly becomes impractical for large matroids.) We illustrate this method with an example.

Example. Let M be the *rank*-3 *whirl*, which is illustrated in Figure 19.

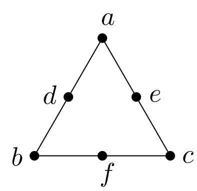

Figure 19: The rank-3 whirl.

Let A be a matrix over the field $\mathbb{F}$ such that $M = M[I|A]$. Assume that A has the following form:

$$A = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} d & e & f \end{array} \\ \left[\begin{array}{ccc} & & \\ & & \\ & & \end{array} \right] \end{array}$$

Let $X = \{a, b, c\}$. Then $C(d, X) = \{a, b, d\}$, $C(e, X) = \{a, c, e\}$, and $C(f, X) = \{b, c, f\}$. By applying Proposition 7.1, we see that A can be rewritten as follows, where each entry denoted $*$ is non-zero.

$$A = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} d & e & f \end{array} \\ \left[\begin{array}{ccc} * & * & 0 \\ * & 0 & * \\ 0 & * & * \end{array} \right] \end{array}$$

Therefore $G(A)$ is the graph shown in Figure 20.

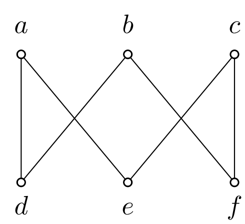

Figure 20: $G(A)$.

We scale every edge in the maximal forest $\{ad, ae, bd, bf, ce\}$ to obtain the following matrix.

$$A = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} d & e & f \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & x_1 \end{bmatrix} \end{array}$$

Now we know that x_1 must be non-zero. Other information about x_1 comes from evaluating determinants. For example, since $\{d, e, f\}$ is a basis, the determinant of $A[\{a, b, c\}, \{d, e, f\}]$ must be non-zero. This determinant is equal to $1 - x_1$, so this shows that $x_1 \neq 1$. In fact, if we evaluate any subdeterminant at all, we will arrive at one of the inequations $x_1 \neq 0$ or $x_1 \neq 1$. (We may also arrive at something trivially true, such as $1 \neq 0$.)

Therefore, if $\mathbb{F}$ is any field with more than two elements, we can set x_1 be a number in $\mathbb{F} - \{0, 1\}$, and then A represents M. On the other hand, in GF(2), there are no numbers other than zero and one, so our argument shows that M cannot be represented over GF(2). $\diamondsuit$

In the next example, we construct a matroid that is not representable over any field.

Definition 7.3. Consider the sparse paving rank-four matroid on the ground set $\{1, \ldots, 8\}$ with the collection

$$\{1, 2, 5, 6\}, \{1, 4, 5, 8\}, \{2, 3, 6, 7\}, \{3, 4, 7, 8\}, \{2, 4, 6, 8\}$$

as its set of non-spanning circuits. (Therefore its bases are all the four-element subsets of $\{1, \ldots, 8\}$ other than the five listed here.) This matroid is known as the *Vámos matroid*, and is denoted by V_8. Figure 21 shows a geometric representation of V_8.

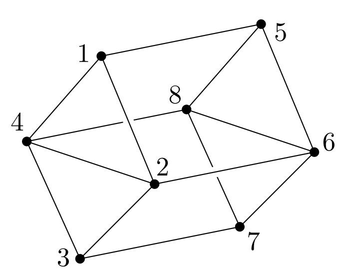

Figure 21: The Vámos matroid.

Proposition 7.4. *The Vámos matroid is not representable over any field.*

Proof. Let $X = \{1, 2, 3, 6\}$, and let $Y = \{4, 5, 7, 8\}$. Then X is a basis of V_8. Assume that the matrix A has its rows labelled by X and its columns labelled by Y, and that A represents V_8 over some field $\mathbb{F}$. By Proposition 7.1 we deduce that A must have the structure

$$\begin{array}{c c} & \begin{array}{cccc} 4 & 5 & 7 & 8 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 6 \end{array} & \begin{bmatrix} * & * & 0 & * \\ * & * & * & * \\ * & 0 & * & * \\ * & * & * & * \end{bmatrix} \end{array}$$

where the $*$ symbols represent non-zero elements of $\mathbb{F}$.

Now the set of all edges incident with 6 along with all the edges incident with 8 form a spanning tree in the graph $G(A)$. By applying Proposition 7.2 and scaling, we can assume that

$$A = \begin{array}{c c} & \begin{array}{cccc} 4 & 5 & 7 & 8 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 6 \end{array} & \begin{bmatrix} x_1 & x_2 & 0 & 1 \\ x_3 & x_4 & x_5 & 1 \\ x_6 & 0 & x_7 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

where $x_1, \ldots, x_7$ are variables representing non-zero elements of $\mathbb{F}$.

The fact that $\{2, 4, 6, 8\}$ is a circuit means that the determinant of $A[\{1, 3\}, \{4, 8\}]$ is zero. This determinant is $x_1 - x_6$, so $x_6 = x_1$. Thus we replace every occurrence of x_6 with x_1. Similarly, $\{1, 4, 5, 8\}$ is a circuit, so the determinant of $A[\{2, 3, 6\}, \{4, 5, 8\}]$ is zero. Thus

$$\begin{vmatrix} x_3 & x_4 & 1 \\ x_1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = x_1 - x_3 + x_4 - x_1x_4 = 0.$$

Next, the fact that $\{3, 4, 7, 8\}$ is a circuit implies that $A[\{1, 2, 6\}, \{4, 7, 8\}]$ has zero determinant. Therefore

$$\begin{vmatrix} x_1 & 0 & 1 \\ x_3 & x_5 & 1 \\ 1 & 1 & 1 \end{vmatrix} = x_3 - x_1 - x_5 + x_1x_5 = 0.$$

By adding these expressions together we see that

$$0 = x_4 - x_5 - x_1x_4 + x_1x_5 = (x_4 - x_5) - x_1(x_4 - x_5).$$

Assume that $x_4 - x_5 \neq 0$. Then $x_1 = (x_4 - x_5)(x_4 - x_5)^{-1} = 1$. This means that $A[\{1,6\},\{4,8\}]$ has zero determinant, which implies that $\{2,3,4,8\}$ is dependent. But this is not true, so we assume that $x_4 - x_5 = 0$. Now $A[\{2,6\},\{5,7\}]$ has zero determinant, which means that $\{1,3,5,7\}$ is dependent. This is not true, so we have a contradiction that completes the proof. □

Equivalence of representations. Assume A is a matrix with entries from the field $\mathbb{F}$, and that the rows and columns of A are labelled by X and Y respectively. Recall that if $x \in X$ and $y \in Y$ are such that the entry A_{xy} is non-zero, then we can pivot on A_{xy}. This operation produces a matrix A^{xy}, with rows labelled by $(X - x) \cup y$, and columns labelled by $(Y - y) \cup x$. Moreover, pivoting does not change the corresponding matroid. That is, $M[I|A] = M[I|A^{xy}]$. In addition to the pivoting operation, we can scale (that is, multiply) the rows and columns of A by non-zero elements of $\mathbb{F}$. The resulting matrix represents the same matroid.

Definition 7.5. Suppose that A and A' are matrices over the field $\mathbb{F}$ with rows and columns labelled by the elements in $X \cup Y$, and assume that $M[I|A] = M[I|A']$. If A' can be obtained from A by pivoting, by scaling rows and columns with non-zero elements of $\mathbb{F}$, and by applying automorphisms of the field $\mathbb{F}$ to all the entries of the matrix, then we say that A and A' are *algebraically equivalent*.

There is also a notion of *geometric* equivalence. In this case, we are not allowed to apply automorphisms of the field $\mathbb{F}$; that is, if A' and A are geometrically equivalent, then A' can be obtained from A by pivoting, and scaling rows and columns.

Definition 7.6. Suppose that M is an $\mathbb{F}$-representable matroid and that if A and A' are matrices such that $M = M[I|A] = M[I|A']$, then A and A' are algebraically equivalent. Then we say that M is *uniquely representable* over the field $\mathbb{F}$.

Recall that a matroid is binary (or ternary) if it is representable over GF(2) (or GF(3), respectively).

Lemma 7.7. *Every binary matroid is uniquely representable over the field* GF(2).

Proof. Let M be a binary matroid, and suppose that A and A' are matrices over GF(2) such that $M = M[I|A] = M[I|A']$. By pivoting on entries of

A', we can assume that the rows of A and A' are labelled by the set X, and that the columns of A and A' are labelled by Y, where X is a basis of M.

If $x \in X$ and $y \in Y$, then by Proposition 7.1, A_{xy} is non-zero if and only if x is in the circuit $C(y, X)$ that is contained in $X \cup y$. But this is true if and only if A'_{xy} is non-zero. So an entry in A is non-zero if and only if the corresponding entry in A' is non-zero. But A and A' are matrices over the field GF(2), which contains a unique non-zero element. Therefore A and A' must in fact be equal. By pivoting, we have transformed A' into A, so A and A' are geometrically equivalent. □

In fact we can strengthen this result further. The next result, along with Proposition 7.2, implies that a binary matroid is uniquely representable over *any* field it can be represented over.

Lemma 7.8. *Let M be a binary matroid. Let A and A' be matrices over the field $\mathbb{F}$ with the same row and column labels, and assume that $M = M[I|A] = M[I|A']$. Let S be the set of edges of a maximal forest in $G(A) = G(A')$. Assume that $A_e = A'_e = 1$ for every edge $e \in S$. Then $A = A'$, and every entry of A or A' is 1, 0, or -1.*

Proof. First note that Proposition 7.1 implies that $G(A) = G(A')$. Let S' be the set of edges in $G(A)$ such that A_e is 1 or -1. Assume that there is an edge, e, not in S'. As S' contains S, there is a cycle, C, of $G(A)$ such that e is in C, and every other edge of C is in S'. We choose e so that C is as small as possible. If there is an edge, x, not in C such that x joins two vertices of C, then x divides C into two cycles that are smaller than C. Exactly one of these cycles contains e. If x is in S', then the cycle that contains e contains a unique edge not in S' (namely e). If x is not in S', then the cycle that does not contain e contains a unique edge not in S' (namely x). In either case, we can choose a cycle that is smaller than C, and our choice is contradicted. Therefore C is an induced cycle of $G(A)$. This means that the submatrix of A containing rows and columns that contain edges of C has the following form (after possibly permuting rows and columns).

$$D = \begin{bmatrix} \pm 1 & 0 & 0 & \cdots & \pm 1 \\ \pm 1 & \pm 1 & 0 & & 0 \\ 0 & \pm 1 & \pm 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_e \end{bmatrix}$$

Call this submatrix D, and assume that D is $n \times n$. We use the following formula for the determinant of D:

$$\det(D) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) D_{1\sigma(1)} D_{2\sigma(2)} \cdots D_{n\sigma(n)}.$$

where the sum is taken over all permutations of $\{1, \ldots, n\}$, and $\operatorname{sgn}(\sigma)$ is either 1 or -1, depending on whether σ can be expressed as the product of an even number of transpositions. It is easy to see that only two permutations will produce a non-zero term in this sum: the identity permutation, and the permutation that takes each $i \in \{2, \ldots, n\}$ to $i-1$ and 1 to n. Therefore

$$\det(D) = \pm A_e \pm 1.$$

Let A_2 be a matrix over $\mathrm{GF}(2)$ with the same row and column labels as A such that $M = M[I|A_2]$. Proposition 7.1 immediately implies that an entry of A_2 is 1 if and only if the entry in the same location of A is non-zero. Taking the submatrix of A_2 with rows and columns containing edges of C, we see that its determinant is zero, by again using the summation formula for the determinant. Since $M[I|A_2] = M[I|A]$, this means that $\det(D)$ must also be zero. Therefore $\pm A_e \pm 1 = 0$, and this implies that $A_e = \pm 1$, which is a contradiction as e is not in S'. We conclude that every edge of $G(A)$ is in S', so every entry of A is 1, 0, or -1, as claimed.

To prove that $A = A'$, we can let S' be the set of edges in $G(A) = G(A')$ such that $A_e = A'_e$ for each $e \in S'$. Thus S' contains S. We argue as before: we choose a smallest-possible cycle that contains a unique edge, e, not in S', and consider the corresponding subdeterminant. This subdeterminant must be zero, as M is binary, and there is a unique value for A_e and A'_e that satisfies this constraint. Therefore $A_e = A'_e$, and we have a contradiction. Thus S' contains every edge of $G(A) = G(A')$. □

We also have uniqueness of representation over $\mathrm{GF}(3)$, although the proof is more difficult.

Lemma 7.9. *Every ternary matroid is uniquely representable over* $\mathrm{GF}(3)$.

We return later to the question of uniqueness of representation for fields that are larger than $\mathrm{GF}(3)$.

Representability of graphic matroids.

Theorem 7.10. *Graphic matroids are representable over every field.*

Proof. Suppose that G is a graph. Let A' be the vertex-edge incidence matrix of G; that is, A' is the matrix whose rows and columns are labelled by the vertices and edges of G, respectively, where the entry in the row labelled v and column labelled e is one if the vertex v is incident with the edge e, and is zero otherwise. Thus every column of A' contains at most two non-zero entries. For every column that contains two non-zero entries, let us arbitrarily change one of them to -1. For every column that contains only one non-zero entry (corresponding to a loop), let us change this entry to zero. Call the resulting matrix A (it is sometimes known as the "signed vertex-edge incidence matrix of G; after removing loops, it describe an orientation of G). We claim that for any field $\mathbb{F}$, if we view A as a matrix over $\mathbb{F}$, then $M[A] = M(G)$.

Let C be a cycle of G. If $C = \{e\}$ is a loop of G, then the column labelled by e is the zero vector, and therefore $\{e\}$ is dependent in $M[A]$. Now we assume that $C = v_0, e_0, v_1, \ldots, v_{t-1}, e_{t-1}, v_t$ for some $t > 1$, where $v_0, \ldots, v_{t-1}$ are pairwise distinct and $v_0 = v_t$. If $i \in \{0, \ldots, t-2\}$, then by multiplying the column labelled by e_i as required, we can assume that $A_{v_i e_i} = 1$, and $A_{v_{i+1} e_i} = -1$. Furthermore, we can assume that $A_{v_{t-1} e_{t-1}} = 1$ and $A_{v_0 e_{t-1}} = -1$. Now the columns labelled by $e_0, \ldots, e_{t-1}$ sum to the zero vector, and are therefore linearly dependent.

Next we assume that $C = \{e_0, \ldots, e_{t-1}\}$ is a circuit of the matroid $M[A]$. Therefore the columns labelled by these edges form a minimal linearly dependent set. If $t - 1 = 0$, then $C = \{e_0\}$ must consist of the zero vector, and hence e_0 is a loop in G. In this case C is dependent in $M(G)$. Henceforth we assume that $t > 1$. This means that if a row contains a non-zero entry in one of the columns $\{e_0, \ldots, e_{t-1}\}$, it contains at least one more non-zero entry in the same set of columns. Equivalently, if a vertex is incident with one of the edges $\{e_0, \ldots, e_{t-1}\}$, it is incident with at least two. Hence every vertex in $G[\{e_0, \ldots, e_{t-1}\}]$ has degree at least two. Since every forest contains a vertex of degree at most one, it now follows that $G[\{e_0, \ldots, e_{t-1}\}]$ is not a forest, and therefore contains a cycle.

We have shown that every circuit of $M(G)$ contains a circuit of $M[A]$, and every circuit of $M[A]$ contains a circuit of $M(G)$. By using the fact that no circuit of $M(G)$ can be properly contained in another, we see that the circuits of $M(G)$ are exactly the circuits of $M[A]$, as desired. $\square$