4 Minors

Suppose that $G = (V, E)$ is a graph, and that $e \in E$ is an edge of G. There are two ways we can remove e from G. Firstly, G *delete* e, denoted by $G\backslash e$, is the graph $(V, E - e)$. In other words, $G\backslash e$ is obtained from G by simply removing the edge e. If e is not a loop, then we obtain G *contract* e, denoted G/e, by removing e and identifying the two vertices e was incident with in G. More formally, if e joins distinct vertices, u and v say, and ϕ is the incident function for G, then we define G/e to be the graph with vertex set $(V - \{u, v\}) \cup w$, for some $w \notin V$, and with edge set $E - e$, where the incidence function $\phi_{G/e}$ of G/e is defined, for each $x \in E - e$, to be

$$\phi_{G/e}(x) = \begin{cases} \phi(x) & \text{if } \phi(x) \cap \{u, v\} = \emptyset \\ (\phi(x) - \{u, v\}) \cup w & \text{if } \phi(x) \cap \{u, v\} \neq \emptyset. \end{cases}$$

Otherwise, when e is a loop of G, we define G/e to be the graph $G\backslash e$. Any graph produced from G by a (possibly empty) sequence of deletions and contractions is a *minor* of G.

Example. Figure 14 shows a graph G with an edge e, along with the graphs produced by deleting and contracting e. $\diamondsuit$

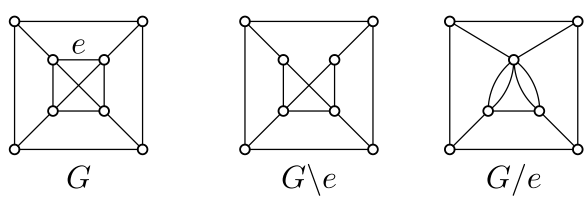

Figure 14: Deleting and contracting an edge e from a graph.

We would like to develop analogous ways of removing elements from a matroid. Let M be a matroid with ground set E and $\mathcal{I}$ as its collection of independent sets. Let e be an element of E. The matroid M *delete* e, denoted $M\backslash e$, is easily defined: its ground set is $E - e$, and its family of independent sets is

$$\{I \subseteq E - e \colon I \in \mathcal{I}\}. \tag{4.1}$$

If e is a loop of M, then M *contract* e, written M/e, is defined to be the same as $M\backslash e$. Otherwise, M/e is defined so that its ground set is $E - e$ and its family of independent sets is

$$\{I \subseteq E - e \colon I \cup e \in \mathcal{I}\}. \tag{4.2}$$

Exercise 4.1. Prove that the collections in (4.1) and (4.2) satisfy conditions **I1**, **I2**, and **I3**.

Exercise 4.2. Let G be a graph, and let e be an edge of G. Using the definition of $M(G)$, as given in Theorem 2.13, prove that $M(G)\backslash e = M(G\backslash e)$.

A matroid produced from M by deleting an element is sometimes called a *single-element deletion* of M. Similarly, a *single-element contraction* of M is produced by contracting an element from M. Any matroid produced from M by a (possibly empty) sequence of deletions and contractions is known as a *minor* of M.

Geometrically, deleting an element from a matroid is the same as removing it from the geometric representation. Remember that we usually do not draw two-point lines in geometric representations. So if we delete a point that is on a three-point line, then we will probably omit that line in the resulting geometric configuration. After deletion, we may wish to move points around to make a more aesthetically pleasing drawing. The following illustration shows a rank-3 matroid M, and a drawing of the deletion $M\backslash d$.

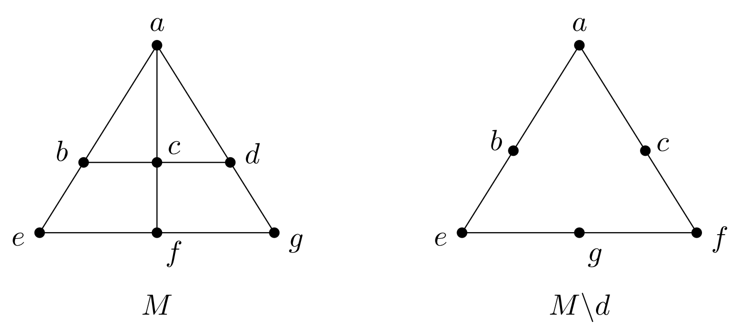

Figure 15: Deleting an element d from a matroid M.

Geometrically, a single-element contraction is a bit more complicated. To construct a geometric representation of M/e from a geometric representation of M, we choose a hyperplane of M that does not contain e. Then we project every point onto that hyperplane from e. This means that for every point x not equal to e, we look at the line containing e and x, and we project x onto the point where that line meets the hyperplane. Notice that this means every point on the hyperplane stays in its current location. Figure 16 gives geometric illustrations of deletion and contraction.

Exercise 4.3. Let e be an element of the matroid M. Prove that every independent set of M/e is also independent in $M\backslash e$.

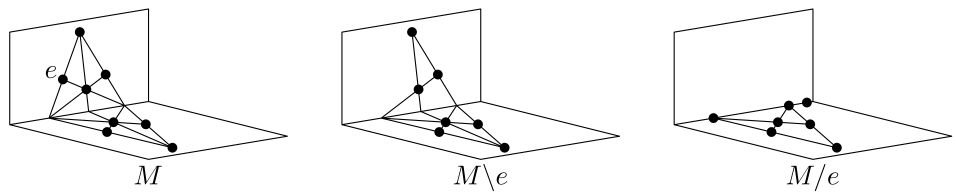

Figure 16: Deleting/contracting an element e from a matroid.

Recall that a coloop of M is a loop of M^*.

Proposition 4.4. *If e is a coloop of the matroid M, then $M\backslash e = M/e$.*

Proof. Let e be a coloop of M. Suppose that I is independent in $M\backslash e$. We claim that I is also independent in M/e. If $I \cup e$ is dependent in M, then Proposition 1.12 implies that $I \cup e$ contains a unique circuit, and this circuit contains e. But this contradicts Proposition 3.7, as e is a coloop. Therefore $I \cup e$ is independent in M, so I is independent in M/e, as claimed. Thus every independent set in $M\backslash e$ is independent in M/e.

From Exercise 4.3, the converse also holds; that is, each independent set of M/e is independent in $M\backslash e$. Since the independent sets of $M\backslash e$ and M/e coincide, the matroids $M\backslash e$ and M/e are equal. $\square$

Our next job is to show that the matroidal operations of deletion and contraction genuinely correspond with the graph operations.

Proposition 4.5. *Let G be a graph, and let e be an edge of G. Then*

(i) $M(G)\backslash e = M(G\backslash e)$, *and*

(ii) $M(G)/e = M(G/e)$.

Proof. The proof of statement (i) is left as Exercise 4.2. We prove (ii). The element e is a loop of $M(G)$ if and only if it is a loop in the graph G. In this case, the result follows immediately from the definitions and (i). Now we assume that e is not a loop, so that e joins distinct vertices u and v. Let w be the vertex of G/e that is produced by identifying u and v.

Let the edge set of G be E. Assume that $X \subseteq E - e$ is dependent in $M(G/e)$. Thus $(G/e)[X]$ contains a cycle $v_0, e_0, v_1, e_1, \ldots, e_{t-1}, v_t$. If $w \notin \{v_0, \ldots, v_{t-1}\}$ then $v_0, e_0, \ldots, e_{t-1}, v_t$ is also a cycle in G. In this case $X \cup e$ is dependent in $M(G)$, so X is dependent in $M(G)/e$. Therefore let us assume (by relabelling as necessary) that $w = v_1$, so that e_0 and e_1 are incident with either u or v in G. If both e_0 and e_1 are incident with u in G, then $v_0, e_0, u, e_1, v_2, e_2, \ldots, e_{t-1}, v_t$ is a cycle of G. Similarly, if both e_0 and e_1 are incident with v, then X contains a cycle of G. Therefore we assume that e_0 is incident with u, and e_1 is incident with v. Then $v_0, e_0, u, e, v, e_1, v_2, e_2, \ldots, e_{t-1}, v_t$ is a cycle in G, so $X \cup e$ contains a cycle of G. A similar argument holds when e_0 is incident with v and e_1 is incident with u. In any case, $X \cup e$ is dependent in $M(G)$, so X is dependent in $M(G)/e$.

For the converse, we suppose that X is dependent in $M(G)/e$, so that $X \cup e$ is dependent in $M(G)$. Thus $G[X \cup e]$ contains a cycle $v_0, e_0, v_1, e_1, \ldots, e_{t-1}, v_t$. Let us assume that e is an edge of a cycle in $G[X \cup e]$. By relabelling we assume that $e = e_1$. Then $v_0, e_0, w, e_2, \ldots, e_{t-1}, v_t$ is a cycle in G/e, so X is dependent in $M(G/e)$. This means that we can assume that e is not contained in any cycle of $G[X \cup e]$. If u and v are both contained in the cycle $v_0, e_0, v_1, e_1, \ldots, e_{t-1}, v_t$, then e is contained in a cycle of $G[X \cup e]$, contrary to our assumption. Therefore at most of one of u and v is contained in $v_0, e_0, v_1, e_1, \ldots, e_{t-1}, v_t$, so we assume that exactly one is. By relabelling we assume that $u = v_1$. Then $v_0, e_0, w, e_1, \ldots, e_{t-1}, v_t$ is a cycle of G/e. Now we assume that $v_0, e_0, v_1, e_1, \ldots, e_{t-1}, v_t$ contains neither u nor v. Then $v_0, e_0, v_1, e_1, \ldots, e_{t-1}, v_t$ is a cycle of G/e. In any case $(G/e)[X]$ contains a cycle, so X is dependent in $M(G/e)$.

We have shown that a set is dependent in $M(G)/e$ if and only if it is dependent in $M(G/e)$, so these two matroids are identical. □

Minor-closed classes. Proposition 4.5 has the following consequence:

Corollary 4.6. *Every minor of a graphic matroid is also graphic.*

Proof. Suppose that M is a graphic matroid, so that $M = M(G)$ for some graph G. If e is an element of $E(M)$, then $M\backslash e = M(G\backslash e)$, so $M\backslash e$ is graphic. Similarly, $M/e = M(G/e)$, so M/e is also graphic. As any minor of M can be obtained by a sequence of deletions and contractions, the result follows by an easy induction argument. □

Definition 4.7. Suppose that $\mathcal{M}$ is a class of matroids. We say that $\mathcal{M}$ is *minor-closed* if, whenever $M \in \mathcal{M}$, every minor of M is also in $\mathcal{M}$.

Thus the class of graphic matroids is minor-closed. The next exercise shows that the class of uniform matroids is minor-closed.

Exercise 4.8. Let M be isomorphic to the uniform matroid $U_{r,n}$, and let e be an element of $E(M)$.

(i) Prove that if $r < n$, then $M\backslash e \cong U_{r,n-1}$, and otherwise $M\backslash e \cong U_{r-1,n-1}$.

(ii) Prove that if $0 < r$, then $M/e \cong U_{r-1,n-1}$, and otherwise $M/e \cong U_{r,n-1}$.

Exercise 4.9. Prove that the class of sparse paving matroids is minor-closed.

Exercise 4.10. Prove that the class of transversal matroids is closed under deleting elements. Prove that it is *not* closed under contracting elements.

Definition 4.11. Let $\mathcal{M}$ be a minor-closed class of matroids. An *excluded minor* for $\mathcal{M}$ is a matroid M, such that M is not in $\mathcal{M}$, but $M\backslash e$ and M/e are in $\mathcal{M}$ for any element $e \in E(M)$.

Exercise 4.12. Let $\mathcal{M}$ be a minor-closed class of matroids. Prove that a matroid M belongs to $\mathcal{M}$ if and only if it does not contain any excluded minor as a minor.

It follows that we can characterise a minor-closed class $\mathcal{M}$ by listing its excluded minors (at least theoretically: the list might be long and complicated). Later, we will see the excluded-minor characterisations of some familar classes. These characterisations are amongst the most famous results in matroid theory.

Minors and bases, circuits, rank. Next we examine how the operations of deletion and contraction affect bases, circuits, and the rank function.

Proposition 4.13. *Let M be a matroid, let $\mathcal{B}$ be its family of bases, and let e be an element in $E(M)$.*

(i) *If e is not a coloop, then*

$$\{B \in \mathcal{B} \colon e \notin B\}$$

is the family of bases of $M\backslash e$.

(ii) *If e is not a loop of M, then*

$$\{B - e \colon B \in \mathcal{B}, e \in B\}$$

is the family of bases of M/e.

Proof. Assume that e is not a coloop of M. Then we can let B be a basis of M that does not contain e, by Proposition 3.7. Certainly B is independent in $M\backslash e$. Moreover, if B is not maximally independent in $M\backslash e$, then B is not maximally independent in M, contradicting the fact that B is a basis of M. Therefore B is indeed a basis of $M\backslash e$.

Now suppose that B is a basis of $M\backslash e$. Then B is independent in M, and B certainly does not contain e. If B is not a basis of M, then $|B| < r(M)$. As e is not a coloop of M, there is a basis B' of M that does not contain e. Now B' is an independent set of $M\backslash e$, and $|B'| = r(M) > |B|$. As B is also an independent set of $M\backslash e$, there is an element $e' \in B' - B$ such that $B \cup e'$ is an independent set of $M\backslash e$. This contradicts the fact that B is maximally independent in $M\backslash e$. Therefore B is indeed a basis of M, with $e \notin B$. This completes the proof of (i).

Next we assume that e is not a loop of M. Suppose that B is a basis of M that contains e. Then $B - e$ is independent in M/e. If $B - e$ is properly contained in an independent set I of M/e, then $I \cup e$ is independent in M, and properly contains B. This is a contradiction, so $B - e$ is indeed a basis of M/e. Now suppose that B' is a basis of M/e. Then $B' \cup e$ is independent in M. If $B' \cup e$ is not a basis of M, then it is properly contained in an independent set I, but then $I - e$ is independent in M/e and properly contains B', which is a contradiction. Therefore $B' \cup e$ is a basis of M. Letting $B = B' \cup e$, we see that B is a basis of M such that $e \in B$, and $B - e$ is a basis of M/e. This completes the proof of (ii). □

Proposition 4.14. *Let M be a matroid whose family of circuits is $\mathcal{C}$, and let e be an element in $E(M)$.*

(i) *The family of circuits of $M\backslash e$ is*

$$\{C \in \mathcal{C} \colon e \notin C\}.$$

(ii) *If e is not a loop, the circuits of M/e are the minimal members of*

$$\{C - e \colon C \in \mathcal{C}\}.$$

Proof. Suppose that C is a circuit of M that does not contain e. Then C is dependent in $M\backslash e$, but all its proper subsets are independent, so it is a circuit of $M\backslash e$. On the other hand, a circuit of $M\backslash e$ is dependent in M, but all its proper subsets are independent in M, so it is a circuit of M that does not contain e. This proves (i).

For (ii), let C' be a circuit of M/e. Then $C' \cup e$ is dependent in M, so it contains a circuit of M. This implies that C' contains, as a subset, a member of $\{C - e\colon C \in \mathcal{C}\}$. Assume that C' properly contains such a set C_1, so $C_1 \cup e$ is dependent in M. Let x be an element in $C' - C_1$. Then $(C' - x) \cup e$ is dependent in M, as it contains $C_1 \cup e$. But this means that $C' - x$ is dependent in M/e, a contradiction as C' is minimally dependent in M/e. We have now proved that every circuit of M/e is a member of $\{C - e\colon C \in \mathcal{C}\}$. In fact, since no circuit of M/e properly contains another (by **C2**), every circuit of M/e is a minimal member of $\{C - e\colon C \in \mathcal{C}\}$.

Now let C' be a minimal member of $\{C - e\colon C \in \mathcal{C}\}$. This means that either C' or $C' \cup e$ is a circuit of M; in either case, $C' \cup e$ contains a circuit of M, so C' is dependent in M/e. Hence C' contains a circuit C_1 of M/e. By the previous paragraph, C_1 is a member of $\{C - e\colon C \in \mathcal{C}\}$. As $C_1 \subseteq C'$ and C' is a minimal member of this set, it follows that $C' = C_1$, and hence C' is a circuit of M/e.

We have now shown that the circuits of M/e are precisely the minimal members of $\{C - e\colon C \in \mathcal{C}\}$. $\square$

Exercise 4.15. Let C be a circuit of the matroid M, and let e be an element not in C. Prove that C is a union of circuits in M/e.

Recall that for matroids M and N, we use r_M and r_N to refer to their respective rank functions.

Proposition 4.16. *Let M be a matroid, and let e be an element of $E(M)$. Then $r_{M\backslash e}(X) = r_M(X)$ and $r_{M/e}(X) = r_M(X \cup e) - r_M(\{e\})$, for every subset $X \subseteq E(M) - e$.*

Proof. Let I be a maximum-sized independent set of $M\backslash e$ contained in X. Then I is independent in M, and no larger independent set of M is contained in X, for such a set would also be independent in $M\backslash e$. Similarly, a maximum-sized independent set of M contained in X is also maximum-sized in $M\backslash e$. Thus $r_{M\backslash e}(X) = r_M(X)$.

Suppose e is a loop. Then $M/e = M\backslash e$. But in this case $r_M(\{e\}) = 0$ and $r_M(X \cup e) = r_M(X)$ for any subset $X \subseteq E(M) - e$. Therefore $r_M(X \cup e) - r_M(\{e\}) = r_M(X)$, which, as shown in the previous paragraph, is equal to $r_{M\backslash e}(X)$. Therefore the result holds in the case that e is a loop.

Now we may now assume that $r_M(\{e\}) = 1$. Let I be a maximum-sized independent set of M/e contained in X, so that $r_{M/e}(X) = |I|$. Then $I \cup e$ is independent in M. If there is a larger independent set of M contained in $X \cup e$, then we could extend $I \cup e$ using **I3**, and produce an independent set of M that properly contains $I \cup e$ and is contained in $X \cup e$. This would imply the existence of an independent set of M/e that is contained in X and is larger than I. This is a contradiction, so

$$r_M(X \cup e) = |I \cup e| = |I| + 1 = r_{M/e}(X) + r_M(\{e\}).$$

The result follows. □

Minors and duality.

Exercise 4.17. Consider a planar embedding of a graph G. Let G^* be the geometric dual of this drawing. Let e be a non-loop edge in G. Prove that the graph we produce from the planar embedding of G by deleting e and then taking the geometric dual is the same as the graph we produce by contracting e from G^*.

This previous exercise suggests an attractive duality relationship between deletion and contraction. The next proposition shows that this relationship applies for all matroids, and not merely the cycle matroids of planar graphs.

Proposition 4.18. *Let M be a matroid, and suppose that e is an element of $E(M)$.*

(i) $(M/e)^* = M^* \backslash e$, *and*

(ii) $(M \backslash e)^* = M^*/e$.

Proof. Let $E = E(M)$. First we prove the statement (i) holds when e is not a loop. Suppose that e is not a loop of M, and let B^* be a subset of $E - e$. We have the following sequence of equivalent statements.

$$\begin{aligned}
&B^* \text{ is a basis of } (M/e)^* \\
\Leftrightarrow\quad &(E - e) - B^* \text{ is a basis of } M/e \\
\Leftrightarrow\quad &((E - e) - B^*) \cup e \text{ is a basis of } M \\
\Leftrightarrow\quad &E - B^* \text{ is a basis of } M \text{ containing } e \\
\Leftrightarrow\quad &B^* \text{ is a basis of } M^* \text{ not containing } e \\
\Leftrightarrow\quad &B^* \text{ is a basis of } M^* \backslash e
\end{aligned}$$

Note, in particular, that the statements on the second and third lines are equivalent since e is not a loop of M, using Proposition 4.13(ii); and the statements on the second-to-last and last lines are equivalent since e is not a coloop of M^*, using Proposition 4.13(i). As $(M/e)^*$ and $M^*\backslash e$ have the same ground set and the same family of bases, $(M/e)^* = M^*\backslash e$. Thus, we have proved that statement (i) is true when e is not a loop of M.

By applying this result to M^* we see that if e is not a loop of M^*, then

$$(M^*/e)^* = (M^*)^*\backslash e = M\backslash e. \tag{4.3}$$

Now we assume that e is a loop of M. Then $M/e = M\backslash e$ by the definition of contraction. This implies that $(M/e)^* = (M\backslash e)^*$. Because $\{e\}$ is a circuit of M, the element e is not a member of any basis of M. Therefore e is not a coloop of M by Proposition 3.7, and hence e is not a loop in M^*. Therefore Equation (4.3) says that $M\backslash e = (M^*/e)^*$. Hence $(M\backslash e)^* = ((M^*/e)^*)^* = M^*/e$. We have shown that $(M/e)^* = (M\backslash e)^*$ and $(M\backslash e)^* = M^*/e$, so $(M/e)^* = M^*/e$. But e is a coloop of M^*, so $M^*/e = M^*\backslash e$, by Proposition 4.4. Therefore $(M/e)^* = M^*\backslash e$, so statement (i) is also true when e is a loop of M.

Now $M\backslash e = (M^*)^*\backslash e = (M^*/e)^*$, by statement (i). Therefore $(M\backslash e)^* = ((M^*/e)^*)^* = M^*/e$, so we have also proved statement (ii). □

We also restate Proposition 4.18 in the following useful form.

Corollary 4.19. *Let M be a matroid, and suppose that e is an element of $E(M)$. Then*

(i) $M\backslash e = (M^*/e)^*$, *and*

(ii) $M/e = (M^*\backslash e)^*$.

Exercise 4.20. Let M be a matroid. Prove that if N is a minor of M, then N^* is a minor of M^*.

Deleting and contracting sets. Next we show that when deleting and contracting multiple elements from a matroid, the order does not matter.

Proposition 4.21. *Let M be a matroid, and let e and f be two distinct elements in $E(M)$. Then*

(i) $(M\backslash e)\backslash f = (M\backslash f)\backslash e$.

(ii) $(M/e)/f = (M/f)/e$.

(iii) $(M\backslash e)/f = (M/f)\backslash e$.

Proof. It is easy to see that I is an independent set of $(M\backslash e)\backslash f$ if and only if $I \subseteq E(M) - \{e, f\}$ and I is independent in M. But this is true if and only if I is independent in $(M\backslash f)\backslash e$, so we have proved the first statement.

Now we consider statement (ii). By applying Proposition 4.18 and Corollary 4.19 we deduce that

$$(M/e)/f = (M^*\backslash e)^*/f = ((M^*\backslash e)\backslash f)^*.$$

By statement (i) this is equal to $((M^*\backslash f)\backslash e)^*$. Now by again applying Proposition 4.18 and Corollary 4.19, we see that

$$((M^*\backslash f)\backslash e)^* = (M^*\backslash f)^*/e = (M/f)/e.$$

This completes the proof of statement (ii). Now we consider the third statement. If f is a loop of M, then it is a loop of $M\backslash e$, so $(M\backslash e)/f$ is equal to $(M\backslash e)\backslash f$. By the first statement this is equal to $(M\backslash f)\backslash e$. As f is a loop of M, this is $(M/f)\backslash e$, as required.

Now we assume that f is not a loop of M. It is therefore not a loop of $M\backslash e$. Let I be a subset of $E(M) - \{e, f\}$. The definitions of contraction and deletion imply that I is independent in $(M\backslash e)/f$ if and only if $I \cup f$ is independent in $M\backslash e$, which is true if and only if $I \cup f$ is independent in M. This is equivalent to I being independent in M/f, which is true if and only if I is independent in $(M/f)\backslash e$. Therefore we have proved the third statement. $\square$

By virtue of the last result, we can remove the parentheses from expressions such as $(M\backslash e)\backslash f$ without any ambiguity. Proposition 4.21 also means that we can sensibly talk about deleting or contracting a set of elements, without specifying the order in which we delete or contract them. In other words, when M is a matroid, and X and Y are disjoint subsets of $E(M)$, the matroid $M/X\backslash Y$ is well defined: it is the matroid produced by contracting the elements of X and deleting the elements of Y, in any order.

Let M be a matroid on the ground set E. If X is some subset of E, then $M|X$, the *restriction* of M to X, is defined to be $M\backslash(E - X)$. Now $M|X$ is a matroid, and its independent sets are exactly $\{I \subseteq X : I \in \mathcal{I}(M)\}$. Therefore the bases of $M|X$ are the maximum-sized independent sets of M that are contained in X. Henceforth we will call such a set a *basis of* X. The matroid $M/(E-X)$ is sometimes called the *contraction* or the *co-restriction* of M to X, and is denoted $M.X$.

Proposition 4.22. *Let M be a matroid, and let X be a subset of $E(M)$. If B is a basis of $M|X$, then $M/X = M/B\backslash(X - B)$.*

Proof. The proof is by induction on $r(X)$. Suppose $r(X) = 0$. Then every element in X is a loop. Since $M/e = M\backslash e$ for any loop e, it follows by easy induction that $M/X = M\backslash X$. Since the only basis of X is the empty set, this establishes the result in this case.

Now we assume that $r(X) > 0$, and that the result holds for all sets with rank less than $r(X)$. Let B be a basis of X, and let x be an element in B. Proposition 4.16 implies that

$$r_{M/x}(X - x) = r_M(X) - 1 = |B| - 1 = |B - x|.$$

Since $B - x \subseteq X - x$ and $B - x$ is independent in M/x, it follows that $B - x$ is a basis of $X - x$ in M/x. By induction, we see that $(M/x)/(X - x) = (M/x)/(B - x)\backslash(X - B) = M/B\backslash(B - X)$. This completes the proof. $\square$

This has the following corollary.

Corollary 4.23. *Let N be a minor of the matroid M. Then N can be expressed as $M/X\backslash Y$, where X and Y are disjoint subsets of $E(M)$, the set X is independent in M, and the set Y is coindependent in M.*

The next result has an easy proof, so we omit it.

Proposition 4.24. *Let M be a matroid with ground set E, and let X and Y be disjoint subsets of E.*

(i) *The independent sets of $M/X\backslash Y$ are the subsets $I \subseteq E - (X \cup Y)$ such that $I \cup B_X$ is an independent set of M whenever B_X is a basis of X.*

(ii) *The bases of $M/X\backslash Y$ are the subsets $B \subseteq E - (X \cup Y)$ such that $B \cup B_X$ is a basis of M whenever B_X is a basis of X.*

(iii) *$r_{M/X\backslash Y}(Z) = r_M(Z \cup X) - r_M(X)$ for any subset $Z \subseteq E - (X \cup Y)$.*

Exercise 4.25. Prove Proposition 4.24.

Exercise 4.26. Let X and Y be disjoint subsets in the matroid M. Characterise the circuits of $M/X\backslash Y$.

Minors and $\mathbb{F}$-representability. We complete this chapter by showing that the class of $\mathbb{F}$-representable matroids is minor-closed, for any field $\mathbb{F}$.

Lemma 4.27. *Let $\mathbb{F}$ be a field, and suppose that M is an $\mathbb{F}$-representable matroid. Any minor of M is also $\mathbb{F}$-representable.*

Proof. It suffices to prove that any single-element deletion or contraction of M is $\mathbb{F}$-representable, since then the result will follow by induction. If M is $\mathbb{F}$-representable, then there is a collection V of vectors from a vector space $\mathbb{F}^n$ such that the independent sets of M are exactly the linearly independent subsets of V. Let $v \in V$ be an element of the ground set of M. Then the independent sets $M\backslash v$ are exactly the linearly independent subsets of $V - v$.

This demonstrates that the single-element deletion of M is also $\mathbb{F}$-representable. But M^* is $\mathbb{F}$-representable, by Proposition 3.15, so $M^*\backslash e$ is also $\mathbb{F}$-representable, by the previous argument. Therefore $(M^*\backslash e)^* = M/e$ is $\mathbb{F}$-representable, as required. □

This last result illustrates a general principle: if a class of matroids is closed under duality and single-element deletions, then it is also closed under minors.

Now we know that if N is a minor of M, and M is $\mathbb{F}$-representable, then N is $\mathbb{F}$-representable. But we would like to know how to find a representation of N over the field $\mathbb{F}$, given such a representation of M. Suppose that $M = M[I|A]$, where A is a matrix with entries from the field $\mathbb{F}$, and the rows and columns of A are labelled by X and Y respectively. Thus the ground set of M is $X \cup Y$. If y is an element of Y, then it is easy to find a representation of $M\backslash y$ by simply deleting the column labelled by y from A. To contract an element x of X, we first recall that $M/x = (M^*\backslash x)^*$. Now M^* is represented by A^T, and x labels a column of A^T. Therefore we delete the column labelled by x, and take the transpose again to find a representation of M/x. Naturally, this representation is more simply found by deleting the row corresponding to x from A.

It remains only to find representations for $M\backslash x$ and M/y, where $x \in X$ and $y \in Y$. To accomplish this, we need to be able to swap rows and columns. The operation we use to do this is called *pivoting*. Pivoting is not complicated: we consider a matrix of the form $[I|A]$, so that it has an $r \times r$ identity matrix in the first r rows. We want to swap a column x in the identity matrix with a column y in A. This involves simply performing row operations until y is a standard basis vector.

We can also think of pivoting in the following way. Let A_{xy} denote the entry of A in the row labelled by x and the column labelled by y. Assume that $A_{xy} \neq 0$. By permuting rows and columns, we can assume that A_{xy} is in the first row and the first column. Now M is represented over $\mathbb{F}$ by $[I|A]$, where $[I|A]$ has the following form:

$$\left[\begin{array}{c|c|c} & A_{xy} & \mathbf{c}^T \\ \cline{2-3} I_r & \mathbf{d} & A' \end{array}\right]$$

Here $\mathbf{c}$ and $\mathbf{d}$ are column vectors, and A' is the matrix obtained from A by deleting the first row and column. Thus the first r columns of this matrix are labelled by X, and the remaining columns are labelled by Y. We want to perform row operations so that the first column of A contains a one in its first entry, and zeroes elsewhere. To do so, we first multiply the top row by A_{xy}^{-1} (which exists, since $A_{xy} \neq 0$). Then, for each other row, letting α be the first entry in that row of A, we subtract α times the first row from this other row. The result looks like:

$$\left[\begin{array}{c|c|c|c} A_{xy}^{-1} & \mathbf{0}^T & 1 & A_{xy}^{-1}\mathbf{c}^T \\ \hline -A_{xy}^{-1}\mathbf{d} & I_{r-1} & \mathbf{0} & A' - A_{xy}^{-1}\mathbf{d}\mathbf{c}^T \end{array}\right]$$

where $\mathbf{0}$ is the zero vector.

Now if we swap the columns labelled by x and y, we see that M is equal to $M[I|A^{xy}]$, where A^{xy} is the matrix

$$\left[\begin{array}{c|c} A_{xy}^{-1} & A_{xy}^{-1}\mathbf{c}^T \\ \hline -A_{xy}^{-1}\mathbf{d} & A' - A_{xy}^{-1}\mathbf{d}\mathbf{c}^T \end{array}\right].$$

The first row of A^{xy} is labelled by y, while the remaining rows are labelled by $X - x$. Similarly, the first column of A^{xy} is labelled by x, and the remaining columns are labelled by $Y - y$. Now we can easily find representations of $M\backslash x$ or M/y, by deleting the first column or row of A^{xy}, respectively.