3 Duality

The notion of matroid duality encompasses the notion of duality for planar graphs, and orthogonality in linear algebra. It gives us a powerful tool for studying matroids: if it is difficult to examine some particular property of a matroid, then we can always switch to its dual, and see if things are easier to understand.

Before we introduce duality, we make the following definition. Assume B is a basis of a matroid M, and e is in $E(M) - B$. Then $B \cup e$ is certainly dependent, because B is maximally independent. Now Proposition 1.12 asserts that there is a unique circuit contained in $B \cup e$, and that this circuit contains e. We denote this unique circuit by $C(e, B)$, and we call it the *fundamental circuit* of e with respect to B.

Proposition 3.1. *Suppose that B_1 and B_2 are bases of a matroid, and that x is in $B_2 - B_1$. There is an element y in $B_1 - B_2$ such that $(B_1 - y) \cup x$ is a basis.*

Note that this is a genuinely different statement from **B2**, although they look very similar. The axiom **B2** says that if we remove an element in $B_1 - B_2$ from B_1, then we can find an element in $B_2 - B_1$ to replace it so that the resulting set is a basis. Proposition 3.1 says that if we add an element in $B_2 - B_1$ to B_1, then there is an element in $B_1 - B_2$ that we can remove so that the resulting set is a basis.

Proof of Proposition 3.1. Since x is not contained in B_1, by Proposition 1.12 there is a fundamental circuit $C(x, B_1)$, and this circuit contains x. Now $C(x, B_1)$ cannot be contained in B_2, for $C(x, B_1)$ is dependent and B_2 is independent. Therefore there is an element y contained in $C(x, B_1) - B_2$. Then y is contained in $B_1 - B_2$. Since $B_1 \cup x$ contains a unique circuit, and y is contained in this circuit, it follows that $(B_1 - y) \cup x$ contains no circuit, and is therefore independent. As $(B_1 - y) \cup x$ has the same cardinality as B_1, and is independent, it is a basis. This completes the proof. □

Lemma 3.2. *Let M be a matroid with ground set E, and let $\mathcal{B}$ be its family of bases. Define $\mathcal{B}^*$ to be the family of complements of bases; that is,*

$$\mathcal{B}^* = \{E - B \colon B \in \mathcal{B}\}.$$

Then $\mathcal{B}^*$ is the family of bases of a matroid on the ground set E.

Proof. Since $\mathcal{B}$ contains at least one basis, $\mathcal{B}^*$ contains at least one complement of a basis. Therefore $\mathcal{B}^*$ is non-empty, and **B1** is satisfied. Now assume that $E - B_1$ and $E - B_2$ are members of $\mathcal{B}^*$. Let x be an element in

$$(E - B_1) - (E - B_2) = B_2 - B_1.$$

By Proposition 3.1 there is an element y in $B_1 - B_2 = (E - B_2) - (E - B_1)$ such that $(B_1 - y) \cup x$ is a basis. Now $E - ((B_1 - y) \cup x) = ((E - B_1) - x) \cup y$, so $((E - B_1) - x) \cup y$ belongs to $\mathcal{B}^*$. Therefore **B2** is satisfied, and $\mathcal{B}^*$ is indeed a family of bases. □

Definition 3.3. Suppose that M is a matroid on the ground set E, and that $\mathcal{B}$ is its family of bases. Let $\mathcal{B}^*$ be the family of complements of bases. The matroid on the ground set E with $\mathcal{B}^*$ as its family of bases is called the *dual* of M, and is written M^*.

Example. Figure 10 shows a rank-2 matroid and its dual. Check that every basis in the dual is the complement of a basis in the rank-2 matroid. ◊

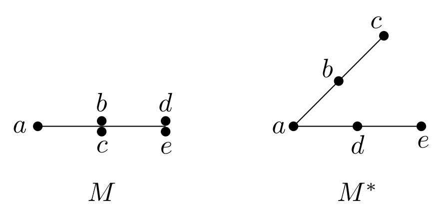


Figure 10: A matroid and its dual

If M is a matroid then obviously $(M^*)^* = M$. Independent sets of M^* are said to be *coindependent sets* of M. Similarly, the bases, and circuits of M^*, are called *cobases* and *cocircuits* of M, respectively. Our next task is to characterise some of these sets. A *spanning* set of a matroid is a subset of the ground set that contains a basis. Therefore a set is spanning if and only if it has the same rank as the entire ground set. The next result follows almost immediately from the definition of the dual matroid.

Proposition 3.4. *Let M be a matroid with ground set E, and let X be a subset of E.*

(i) *The set X is a cobasis of M if and only if $E - X$ is a basis of M.*

(ii) *The set X is coindependent in M if and only if $E - X$ is spanning in M.*

Before we characterise cocircuits we need a definition.

Definition 3.5. Let M be a matroid. A subset $X \subseteq E(M)$ is a *hyperplane* of M if $r(X) = r(M) - 1$, and $r(X \cup x) = r(M)$ for every $x \in E(M) - X$.

Equivalently, a hyperplane is a set that is maximal with respect to being non-spanning. In other words, X is a hyperplane if X is not spanning, but X' *is* spanning whenever $X \subseteq X' \subseteq E(M)$ and $X' \neq X$.

Proposition 3.6. *Let M be a matroid and let X be a subset of $E(M)$. Then X is a cocircuit of M if and only if $E(M) - X$ is a hyperplane of M.*

Proof. We observe that X is a cocircuit if and only if it is dependent in M^*, and it is minimal with respect to this property. This is true if and only if X is minimal with respect to the property that its complement is non-spanning, by Proposition 3.4(ii); that is, if and only if $E(M) - X$ is maximal with respect to being non-spanning, which is equivalent to $E(M) - X$ being a hyperplane. $\square$

A parallel pair of M^* is called a *series pair* of M and a parallel class of M^* is called a *series class* of M. A loop of M^* is known as a *coloop* of M. Thus e is a coloop of M if and only if $\{e\}$ is a cocircuit of M, which (by Proposition 3.6) is true if and only if the complement of $\{e\}$ is a hyperplane in M.

Proposition 3.7. *Let M be a matroid and let e be an element of $E(M)$. The following are equivalent:*

(i) *e is a coloop of M,*

(ii) *e is in every basis of M, and*

(iii) *e is not in any circuit of M.*

Proof. (i) $\Leftrightarrow$ (ii): If e is a coloop of M, then it is a loop of M^*. This means that $\{e\}$ is a circuit of M^*, so e is not contained in any basis of M^*. Since the bases of M^* are the complements of bases of M, it follows that e is in every basis of M. On the other hand, if e is in every basis of M, then it is not in any basis of M^*, and therefore $\{e\}$ is dependent in M^*. This establishes that (i) and (ii) are equivalent.

(ii) $\Leftrightarrow$ (iii): Suppose e is contained in a circuit C. Then $C - e$ is independent in M because it is a proper subset of a minimal dependent set. Thus $C - e$ is contained in a basis B of M. However, B cannot contain e, for then B would contain the circuit C. So if e is contained in a circuit C, then there is a basis B that does not contain e. Conversely, if there is a basis B of M that does not contain e, then $B \cup e$ contains a unique circuit, and this circuit contains e, by Proposition 1.12.

We have shown that e is contained in a circuit if and only if there is a basis that does not contain e. Equivalently, e is not in any circuit if and only if e is in every basis. Therefore (ii) and (iii) are equivalent. □

Now we turn to the rank function. The next result follows immediately from the definition of M^*.

Proposition 3.8. *Let M be a matroid. Then $r(M^*) = |E| - r(M)$.*

If M is a matroid, then r^* denotes the rank function of the dual matroid M^*. Thus $r^*(X)$ is the cardinality of a maximum-sized coindependent set contained in X, for any subset X of $E(M)$.

Proposition 3.9. *Let M be a matroid with ground set E and rank function r. The rank function r^* of M^* is given by the formula*

$$r^*(X) = |X| + r(E - X) - r(M)$$

for any subset $X \subseteq E$.

Proof. Let I^* be a maximum-sized coindependent set contained in X, so $r^*(X) = |I^*|$. Then $E - I^*$ is spanning, by Proposition 3.4. Therefore there is a basis B contained in $E - I^*$. Suppose there is an element x in $X - I^*$ such that x is not in B. Then $I^* \cup x$ is strictly larger than I^*, since $x \notin I^*$, and $I^* \cup x$ is contained in the cobasis $E - B$. Therefore $I^* \cup x$ is coindependent, and contained in X, and this violates our choice of I^*. Therefore $X - I^* \subseteq B$. In particular, since $B \subseteq E - I^*$, we have that $I^* = X - (B \cap X)$.

Observe that $B - X$ is independent. Next, suppose there is an element x in $E - (B \cup X)$ such that $(B - X) \cup x$ is independent. Then $B \cup x$ contains a unique circuit, $C(x, B)$, by Proposition 1.12. Our assumption means that $C(x, B)$ is not contained in $(B - X) \cup x$. Therefore we can choose an element $y \in C(x, B) \cap X$. Since $B \cup x$ contains a unique circuit $C(x, B)$, and y is in $C(x, B)$, it follows that $(B - y) \cup x$ is independent and, as it is the same size as B, it is a basis. As $I^* \cup y$ has empty intersection with $(B - y) \cup x$, we see that $I^* \cup y$ is coindependent. This contradicts our choice of I^*, so $(B - X) \cup x$ is dependent for any element x in $E - (B \cup X)$. Consequently, no independent subset of $E - X$ is larger than $B - X$, for if such a subset existed, **I3** would be violated. This means that $r(E - X) = |B - X|$.

Finally,

$$\begin{aligned} r^*(X) &= |I^*| \\ &= |X| - |B \cap X| \\ &= |X| - (|B| - |B - X|) \\ &= |X| - r(M) + r(E - X) \end{aligned}$$

as required. □

Duality and classes of matroids. Next we will examine how duality affects the various classes of matroids we have already considered. The next result follows immediately from the definition of the dual matroid.

Proposition 3.10. *The dual of a uniform matroid is also uniform. In particular, the dual of $U_{r,n}$ is $U_{n-r,n}$.*

Exercise 3.11. Prove that the dual of a sparse paving matroid is sparse paving.

Exercise 3.12. Construct a transversal matroid whose dual is not transversal.

Our next job is to show that if $\mathbb{F}$ is a field, then the dual of any $\mathbb{F}$-representable matroid is also $\mathbb{F}$-representable. First, we introduce a slightly different way of thinking about representable matroids. Previously, for a matrix A, we found a corresponding matroid $M[A]$ having the columns of A as its ground set. For a representable matroid M with rank r and ground set of size n, it is often convenient to instead obtain a matrix A', such that $[I|A']$ is a representation of M, where A' has r rows and $n-r$ columns, and each element of M corresponds to either a row or column of A'. We formalise this below.

Let A be a matrix with entries from the field $\mathbb{F}$, and suppose that the rows of A are labelled with the ordered set X, while the columns are labeled with the ordered set Y. If $X' \subseteq X$ and $Y' \subseteq Y$, then $A[X', Y']$ denotes the submatrix of A with rows labelled by X' and columns labelled by Y'.

Proposition 3.13. *Let A be a matrix over the field $\mathbb{F}$ with rows and columns labelled by the disjoint ordered sets X and Y respectively, and let $r = |X|$. Let*

$$\mathcal{B} = \{X\} \cup \{Z\colon Z \subseteq X \cup Y,\ |Z| = r,\ \textit{and}\ \det(A[X - Z, Y \cap Z]) \neq 0\}.$$

Then $\mathcal{B}$ is the family of bases of a matroid M on the ground set $X \cup Y$. Moreover, if I is the $r \times r$ identity matrix with columns labelled by X, then $M = M[I|A]$.

Proof. We consider the matrix $[I|A]$ which is formed by appending an $r \times r$ identity matrix in front of A, where we label the first r columns of $[I|A]$ with the r elements of X (in order), and label the remaining columns of $[I|A]$ with the ordered elements of Y. It suffices to show that the bases of $M[I|A]$ are exactly the members of $\mathcal{B}$, as defined in the proposition. To see this, note that a set $Z \subseteq X \cup Y$ is a basis of $M[I|A]$ if and only if $|Z| = r$ and the set of columns labelled by Z is linearly independent. This is true if and only if the $r \times r$ matrix made up of the columns in Z has a non-zero determinant. Let S denote this $r \times r$ submatrix of $[I|A]$. If x is an element in $X \cap Z$, then the column of S labelled by x contains a single non-zero entry. Furthermore, S has a non-zero determinant if and only the matrix produced from S by deleting the column x and the row containing that non-zero entry also has a non-zero determinant. Continuing in this way, we see that S has a non-zero determinant if and only if either $Z = X$ (where we delete rows and columns until we produce the empty matrix), or the matrix $A[X - Z, Y \cap Z]$ has a non-zero determinant. This is true if and only if Z belongs to $\mathcal{B}$. $\square$

When A is a matrix over a field, with rows and columns labelled X and Y respectively, then there is a corresponding matroid with $X \cup Y$ as its ground set, as supplied by Proposition 3.13. Alternatively, we can label the columns of an $|X| \times |X|$ identity matrix I by X, and the columns of A by Y, so that all columns of $[I|A]$ are labelled, and consider the matroid $M[I|A]$ (in the sense of Theorem 2.4). Either way, we are dealing with the same representable matroid.

Example. In Figure 11, A is a matrix over the field GF(3), and a geometric representation of the rank-3 matroid $M[I|A]$ is given. Note that, for example, $A[\{a, b\}, \{d, e\}]$ is

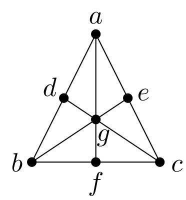
$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

and this matrix has determinant -1. This corresponds to the fact that $\{c, d, e\}$ is a basis of $M[I|A]$. On the other hand, $A[\{a, b\}, \{d, g\}]$ is

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which has determinant equal to 0. This corresponds to the fact that $\{c, d, g\}$ is not a basis.

The matroid shown in Figure 11 is called the *non-Fano* matroid, and is denoted by F_7^-. $\diamondsuit$



$M[I|A]$

$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{cccc} d & e & f & g \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

A

Figure 11: A matrix over GF(3) and the matroid it represents.

Proposition 3.14. *Let M be an $\mathbb{F}$-representable matroid, for some field $\mathbb{F}$, and let B be a basis of M. There is a matrix A over the field $\mathbb{F}$ such that M is equal to $M[I|A]$, where the columns of the identity matrix I are labelled by B.*

Proof. Since M is $\mathbb{F}$-representable, there is a matrix A' over $\mathbb{F}$ such that M is equal to $M[A']$. It is relatively easy to see that if A'' is obtained from A' by a sequence of the following operations, then $M[A''] = M[A']$.

- Multiplying a column by a non-zero constant.
- Multiplying a row by a non-zero constant.
- Adding a row to another row.
- Swapping two columns (as long as we swap the column labels at the same time).

If the column rank of A' is r, then by performing a sequence of these operations (and possibly deleting some rows that contain only zeroes), we can transform A' into a matrix of the form $[I_r|A]$, where I_r is an $r \times r$ identity matrix whose columns are labelled by B. Now it is easy to see that M is equal to $M[I|A]$. $\square$

The last two propositions illustrate that there are two different ways of thinking about an $\mathbb{F}$-representable matroid M: we can take the ground set of M to be the set of columns of a matrix, or the set of rows and columns. We may switch between the two, according to convenience.

Proposition 3.15. *Let $\mathbb{F}$ be a field, and let M be an $\mathbb{F}$-representable matroid. Then M^* is also $\mathbb{F}$-representable. In particular, if A is a matrix over $\mathbb{F}$ such that $M = M[I|A]$, then $M^* = M[I|A^T]$, where A^T is the transpose of A.*

Proof. Let the rows and columns of A be labelled by the (ordered) sets X and Y respectively (so the rows and columns of A^T are labelled by Y and X respectively). Now the bases of M are

$$\{X\} \cup \{Z \colon Z \subseteq X \cup Y,\ |Z| = |X|,\ \det(A[X - Z, Y \cap Z]) \neq 0\}.$$

This means that the bases of M^* are precisely

$$\begin{gathered}\{Y\} \cup \{(X \cup Y) - Z \colon Z \subseteq X \cup Y,\ |Z| = |X|,\ \det(A[X - Z, Y \cap Z]) \neq 0\} = \\ \{Y\} \cup \{Z' \colon Z' \subseteq X \cup Y,\ |Z'| = |Y|,\ \det(A[X \cap Z', Y - Z']) \neq 0\}.\end{gathered}$$

But $A[X \cap Z', Y - Z']$ has non-zero determinant if and only if $A^T[Y - Z', X \cap Z']$ has non-zero determinant, so the bases of M^* are

$$\{Y\} \cup \{Z' \colon Z' \subseteq X \cup Y,\ |Z'| = |Y|,\ \det(A^T[Y - Z', X \cap Z']) \neq 0\}$$

and this means that M^* is exactly $M[I|A^T]$. □

Now we will move to thinking about the duals of graphic matroids. Unlike the case for representable matroids, the dual of a graphic matroid need not be graphic. A matroid whose dual is graphic is said to be *cographic*, and if G is a graph, then $M^*(G)$ denotes the dual of the graphic matroid $M(G)$. Soon we will prove that a cographic matroid might not be graphic. First, we establish some more facts about graphic matroids.

Proposition 3.16. *Suppose that M is a graphic matroid. Then there is a connected graph G such that $M = M(G)$.*

Proof. Since M is graphic, there is a graph G' such that $M = M(G')$. If G' is connected then we are done, so let us assume that G' has connected components $H_1, \ldots, H_t$, where $t > 1$. For each $i \in \{1, \ldots, t\}$, choose a vertex v_i in H_i. We construct the graph G by identifying the t vertices $v_1, \ldots, v_t$. Thus G is certainly connected. Moreover, it is easy to see that a set of edges in G forms a cycle if and only if it forms a cycle in G'. Therefore $M(G)$ has exactly the same family of circuits as $M(G')$. This means that $M(G) = M(G')$, as required. □

The next result follows immediately from Corollary 2.12.

Corollary 3.17. *Let $G = (V, E)$ be a graph with k connected components. The rank of $M(G)$ is $|V| - k$.*

Let G be a connected graph. A *spanning tree* of G is a forest, E', of G such that $G[E']$ is connected and every vertex of G is in $G[E']$.

Exercise 3.18. Let G be a connected graph. Prove that the maximal forests of G are exactly the spanning trees of G.

Let $G = (V, E)$ be a graph, and let X be some subset of E. Then $G\backslash X$ is the subgraph $(V, E - X)$. Note that $G\backslash X$ is not necessarily the same graph as $G[E - X]$. The former has V as its vertex set, whereas the latter contains only those vertices that are incident with at least one edge in $E - X$. An *edge cut* of G is a subset $X \subseteq E$ such that $G\backslash X$ has more connected components than G does. A minimal edge cut is known as a *bond*.

Proposition 3.19. *Let $G = (V, E)$ be a graph. For $X \subseteq E$, the set X is coindependent in $M(G)$ if and only if X is not an edge cut of G.*

Proof. Assume that X is not an edge cut, so that $G\backslash X$ has the same number of components as G. Since G and $G\backslash X$ have the same number of vertices, Corollary 2.12 implies that $M(G)$ and $M(G\backslash X)$ have bases of the same size. In particular, this means that there is a basis of $M(G)$ that is contained in $E - X$. Hence X is coindependent in $M(G)$.

For the converse, assume that X is an edge cut. If G has k components, then $G\backslash X$ has at least $k+1$. Now Corollary 2.12 implies that $r(M(G\backslash X)) < r(M(G))$. This implies that no forest of G can be contained in $E - X$, so no basis of $M(G)$ can be contained in $E - X$. Thus, by Proposition 3.4(ii), X is not coindependent in $M(G)$. □

Since a cocircuit is a minimal set that is not coindependent, and a bond is a minimal edge cut, the next result follows immediately.

Corollary 3.20. *Let $G = (V, E)$ be a graph. For a subset $X \subseteq E$, the set X is a cocircuit of $M(G)$ if and only if X is a bond of G.*

Recall that the degree of the vertex v is

$$|\{e \in E\colon v \in e,\ e \text{ is a non-loop edge}\}| + 2|\{e \in E\colon v \in e,\ e \text{ is a loop}\}|.$$

Exercise 3.21. Prove that the average degree of the graph $G = (V, E)$ is $2|V|/|E|$.

Now we can give an example of a cographic matroid that is not graphic. Recall that K_n is the *complete* graph on n vertices. That is, K_n has exactly n vertices, no loops or parallel edges, and every pair of vertices is joined by an edge. Figure 12 shows the complete graph K_5.

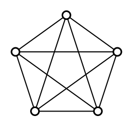


Figure 12: The graph K_5.

Proposition 3.22. *The cographic matroid $M^*(K_5)$ is not graphic.*

Proof. Towards a contradiction, suppose that $M^*(K_5)$ is graphic. By Proposition 3.16, there is a connected graph G such that $M(G) = M^*(K_5)$. Let n be the number of vertices in G. Corollary 3.17 implies that $M(G)$ has rank $n-1$. The number of edges in K_5 is ten, and $r(M(K_5)) = 4$. Thus $M^*(K_5)$ has rank $10-4 = 6$, by Proposition 3.8. Now $n-1 = 6$. As G has ten edges, and seven vertices, the average degree in G is $20/7 < 3$. This means that some vertex v in G has degree at most two, so v is incident with at most two edges. The set of edges incident with v is an edge cut, so this set of edges contains a bond of G. But this means that $M(G) = M^*(K_5)$ contains a cocircuit of size at most two, by Proposition 3.19. As every circuit of $M(K_5)$ has size at least three, this is a contradiction. $\square$

Exercise 3.23. Give an example of a graph with a vertex v such that the set of edges incident with v is not a bond.

Proposition 3.22 doesn't tell the entire story, since some cographic matroids *are* graphic. Recall that, informally, a graph is *planar* if it can be drawn in the plane without any edges crossing.

More formally, a *planar embedding* of G consists of an injective function f_v from $V(G)$ to the plane $\mathbb{R}^2$, and another function f_e taking each edge of G to a "segment": a subset of $\mathbb{R}^2$ that is homeomorphic to the segment $[0, 1]$. The ends of this segment are the image of the ends of the edge, and the images of two distinct edges can only meet at the ends. If we delete the image of these functions from the plane, then we are left with a collection of open discs, and these discs are called the *faces* of the embedding. A face in a planar embedding is (usually) bounded by a cycle of the graph. (Note, however, that a cycle in a planar embedding may bound a collection of several faces, and that a face may be not bounded be a cycle.) Then, a graph is *planar* if it has a planar embedding. A *plane graph* is a graph (the *underlying graph*) together with a planar embedding of the graph. When G is a plane graph, for simplicity we also use G to refer to the underlying graph of G, and the *faces* of G are the faces of the planar embedding of G.

Exercise 3.24. Give an example of a plane graph with a face that is not bounded by a cycle.

Consider a plane graph G. Let F be the set of faces of G. The *geometric dual*, denoted G^*, is a graph with F as its vertex set and E as its edge set. In the planar embedding of G, an edge e is incident with at most two faces, and in G^*, the edge e is incident with exactly these faces. (Note that the geometric dual may depend on our choice of planar embedding.) Figure 13 shows a planar embedding of a graph G, and a geometric dual of G.

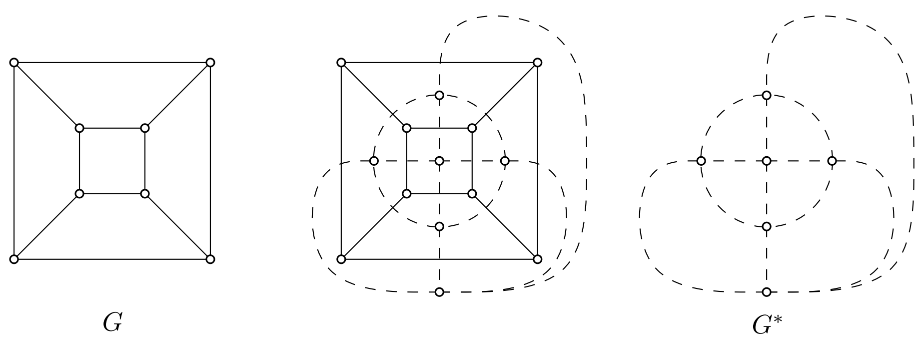


Figure 13: Constructing a geometric dual.

Exercise 3.25. Prove that if every vertex in the graph G has degree at least two, then G contains a cycle.

Theorem 3.26. *Let G be a planar graph. Then $M^*(G)$ is a graphic matroid. In particular, if G is a plane graph with geometric dual G^*, then $M^*(G) = M(G^*)$.*

Proof. Consider a cycle in G^*, and let C be its edge set and F its vertex set. We will show that C contains a bond of G. The vertices of the cycle F

correspond to faces in the planar embedding of G. Using the Jordan Curve Theorem, we see that this cycle of faces divides the vertices of G into two non-empty parts: those 'inside' the cycle, and those 'outside'. Any edge of G that joins these two parts is an edge in C. Thus C is an edge cut of G, and hence contains a bond of G.

Now let X be a bond of G. We will show that X contains a cycle of G^*. Because X is a minimal edge cut, there is a partition (A, B) of the vertices of G such that every edge in X joins a vertex in A to a vertex in B. Moreover, in $G\backslash X$, there is no path from a vertex in A to a vertex in B. Let $\mathcal{F}$ be the set of faces in the planar embedding that are incident with edges in X, and let F be an arbitrary face in $\mathcal{F}$. It is possible that F is the only face incident with an edge in X. In this case $|X| = 1$, and in the geometric dual G^*, the singleton X is a loop. Therefore we will assume that there are at least two faces in $\mathcal{F}$. This implies that every face in $\mathcal{F}$ is bounded by a cycle of G. Since F is incident with an edge of X, it must be incident with at least two, for otherwise we can find a path from a vertex in A to a vertex in B in the graph $G\backslash X$. This shows that in the geometric dual, every face in $\mathcal{F}$ is joined to at least two other faces in $\mathcal{F}$ by edges in X. This in turn implies that the subgraph of G^* with vertex set $\mathcal{F}$ and edge-set X contains a cycle.

Now we have shown that every cycle of G^* contains a bond of G, and every bond of G contains a cycle of G^*. This implies that the cycles of G^* are exactly the bonds of G. The result now follows by Corollary 3.20. $\square$

As we will see later, the converse of Theorem 3.26 also holds. That is, if $M^*(G)$ is graphic, then G is planar.