Lecture 1, exercise 1: Show that $U_{r,n}$ is a matroid.

Recall that for a set E of size n, and an integer r with $0 \leq r \leq n$, we defined $U_{r,n}$ to be the pair $(E, \mathcal{B})$ with $\mathcal{B} = \{B \subseteq E : |B| = r\}$.

Let $E(U_{r,n}) = \{e_1, e_2, \ldots, e_n\}$. We need to show that $\mathcal{B}$ satisfies (B1) and (B2).

For (B1), observe that there is at least one r-element set for any $r \geq 0$ (in particular, if $r = 0$, then $\emptyset$ is such a set). So $\mathcal{B} \neq \emptyset$, and (B1) is satisfied.

It remains to show that (B2) holds. Let $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 - B_2$. Since $|B_1 - B_2| \geq 1$, and $|B_1 - B_2| = |B_2 - B_1|$ (as $|B_1| = |B_2|$), there exists $y \in B_2 - B_1$. Let $B_3 = (B_1 - x) \cup y$. Then $|B_3| = |B_1| = r$, and $B_3 \subseteq E$, so $B_3 \in \mathcal{B}$ as required.

Lecture 2, exercise 1: Let M be a matroid with rank r. Prove that:

> M is uniform if and only if every circuit of M has size $r + 1$.

Let E be the ground set of M, and suppose that M is uniform. Then $M \cong U_{r,n}$ for some non-negative integer r, where $n = |E|$. Every subset of E of size r is a basis, by the definition of a uniform matroid. The independent sets of M are $\mathcal{I} = \{X \subseteq E : X \subseteq B \text{ for some } B \in \mathcal{B}\}$ (by the definition of independent sets), so $\mathcal{I} = \{X \subseteq E : |X| \leq r\}$. A subset of $E(M)$ is dependent if it is not a member of $\mathcal{I}$. So the dependent sets of M are $\{X \subseteq E : |X| \geq r + 1\}$. In particular, the circuits (i.e. the minimal dependent sets) are precisely the subsets of E of size $r + 1$. So every circuit of M has size $r + 1$.

For the other direction, suppose every circuit of M has size $r + 1$. We will first show

$$\mathcal{C}(M) = \{C \subseteq E : |C| = r + 1\};$$

that is, every subset of E with size $r + 1$ is a circuit. This holds vacuously if $|E| = r$, so let $|E| < r$. Towards a contradiction, suppose $X \subseteq E$ with $|X| = r + 1$, but X is not a circuit. If X is dependent, then it is not minimal; that is, it contains a circuit of size at most r, but no such circuit exists. So X is independent. But then M has an independent set of size $r + 1$, so M has a basis of size at least $r + 1$, contradicting that M has rank r. We deduce that every subset of E of size $r + 1$ is a circuit (i.e., a minimal dependent set). Now every subset of E of size r is a maximal independent set (i.e., a basis). Thus $M \cong U_{r,n}$, so M is uniform, as required.

Lecture 2, exercise 2: Let E be a finite set, and let r be an integer with $0 < r < |E|$. Let $\mathcal{C}'$ be a collection of r-element subsets of E such that if C_1 and C_2 are distinct members of $\mathcal{C}'$, then $|C_1 \cap C_2| < r - 1$. Let $\mathcal{B}$ be the family of r-element subsets of E that are not in $\mathcal{C}'$; that is, $\mathcal{B} = \{B \subseteq E : |B| = r \text{ and } B \notin \mathcal{C}'\}$. Prove that $(E, \mathcal{B})$ is a matroid.

We need to prove that $\mathcal{B}$ satisfies (B1) and (B2). Towards a contradiction, suppose that $\mathcal{B} = \emptyset$. Then every r-element subset of E is in $\mathcal{C}'$. Since $r > 0$, we have $r - 1 \geq 0$. We arbitrarily choose a subset X of E with $|X| = r - 1$. As $r < |E|$, we have $r - 1 \leq |E| - 2$, so there are distinct elements $y, z \in E - X$. Then $X \cup y$ and $X \cup z$ are in $\mathcal{C}'$, but $|(X \cup y) \cap (X \cup z)| = |X| = r - 1$, a contradiction. We deduce that $\mathcal{B} \neq \emptyset$, satisfying (B1).

Now for (B2), suppose that $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$. Note that (since $|B_1| = |B_2|$) we have $|B_2 - B_1| = |B_1 - B_2| \geq 1$. If $|B_2 - B_1| = 1$, then $B_2 = (B_1 - x) \cup y$. That is, choosing y as the unique element in $B_2 - B_1$, we have $(B_1 - x) \cup y \in \mathcal{B}$ as desired. So (B2) holds when $|B_2 - B_1| = 1$. Now suppose $|B_2 - B_1| \geq 2$. Then there exist distinct elements $e, f \in B_2 - B_1$. Suppose (towards a contradiction) that $(B_1 - x) \cup e \notin \mathcal{B}$ and $(B_1 - x) \cup f \notin \mathcal{B}$. Then, as $|(B_1 - x) \cup e| = |(B_1 - x) \cup f| = r$, we have $(B_1 - x) \cup e \in \mathcal{C}'$ and $(B_1 - x) \cup f \in \mathcal{C}'$. But $\big|\big((B_1 - x) \cup e\big) \cap \big((B_1 - x) \cup f\big)\big| = |B_1 - x| = r - 1$, contradicting that the intersection of two distinct members of $\mathcal{C}'$ has size smaller than $r - 1$. From this contradiction, we deduce that at least one of $(B_1 - x) \cup e$ and $(B_1 - x) \cup f$ is in $\mathcal{B}$. So (B2) also holds when $|B_2 - B_1| \geq 2$. This completes the proof.