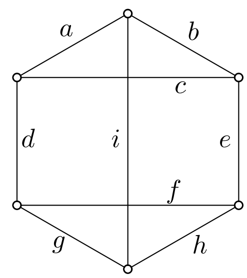
Total marks available: 60

You may use results stated in the course notes, lecture notes, or from previous assignments, but you should cite any results that you use. You should not use any other resources, printed or online. This is not a collaborative assignment – your submitted work should reflect your individual effort.

Q1. Consider the graph G with drawing given below: **[6]**



(a) Draw a planar embedding of G.

(b) Draw the geometric dual corresponding to the embedding given in (a).

(c) Draw a geometric representation of $M^*(G)$.

Q2. Let M be the matroid represented over the field GF(5) by the matrix **[12]**

$$\begin{array}{c} \begin{matrix} a & b & c & d & e & f & g \end{matrix} \\ \begin{bmatrix} 1 & 0 & 2 & 1 & 3 & 2 & 2 \\ 4 & 1 & 0 & 0 & 4 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 & 2 \end{bmatrix} \end{array}$$

(a) Convert this matrix into standard form, where the columns of the identity are labelled by the elements (a, b, c) (in this order).

(b) Draw a geometric representation of M.

(c) Give a matrix that represents M^*. (Be sure to label the columns of this matrix.)

(d) Draw a geometric representation of M^*.

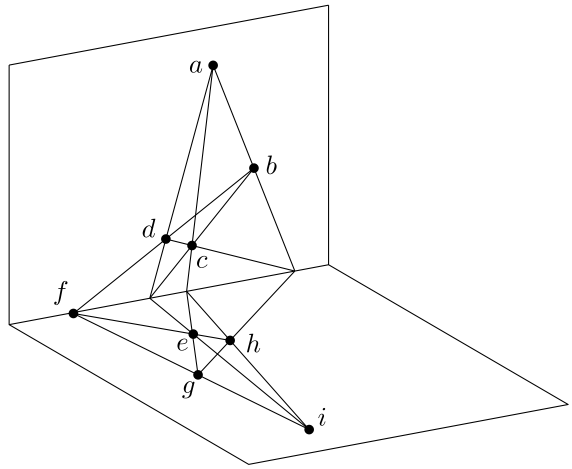
(e) Find a reduced matrix representation of M where the rows are labelled by (d, b, c) and the columns are labelled by (a, e, f, g).

(f) Find reduced matrix representations of M/d and $M\backslash a$.

(g) Find geometric representations of M/d and $M\backslash a$.

Q3. Let M be a matroid on the ground set E with $\mathcal{I}$ as its collection of independent sets. Assume that $e \in E$ is not a loop. Prove that the collection $\{I \subseteq E - e \colon I \cup e \in \mathcal{I}\}$ obeys the axioms **I1**, **I2**, and **I3**. (In other words, show that M/e, as defined in lectures, is a matroid.) **[5]**

Q4. The following is a geometric representation of a rank-4 matroid M.



Draw geometric representations of the following minors of M. **[8]**

(a) M/a,

(b) M/b,

(c) $M/b\backslash c\backslash e$,

(d) M/f.

Q5. Recall that a rank-r matroid is *sparse paving* if every circuit has cardinality at least r, and whenever C and C' are distinct circuits with $|C| = |C'| = r$, then $|C \cap C'| < r - 1$. Prove that the class of sparse paving matroids is closed under taking minors. **[6]**

Q6. Let M be a matroid on the ground set E. A set $Z \subseteq E$ is a *cyclic flat* of M if it is a flat, and any element $z \in Z$ is contained in a circuit C such that $C \subseteq Z$. (Note that this means that if the empty set is a flat, then it is a cyclic flat.)

(i) Let C be a circuit of M. Prove that the closure of C is a cyclic flat. (Hint: Proposition 5.9 may be helpful.)

(ii) Give an example of a matroid containing a cyclic flat F that is not the closure of a circuit, and $|F| \geq 1$.

(iii) Let $X \subseteq E$. Prove that X is dependent in M if and only if there is a cyclic flat Z of M such that $|X \cap Z| > r(Z)$.

(iv) Prove that Z is a cyclic flat of M if and only if $E - Z$ is a cyclic flat of M^*. **[14]**

Q7. Let e be an element of a matroid M. Prove that the following are equivalent:

(i) e is a coloop.

(ii) If $X \subseteq E(M)$ and $e \in \text{cl}(X)$, then $e \in X$.

(iii) $E(M) - e$ is a hyperplane. **[4]**

Q8. Let M be a matroid on the ground set E, and let X be a subset of E. Prove that $\text{cl}(X)$ is equal to the intersection of all flats of M that contain X. **[5]**