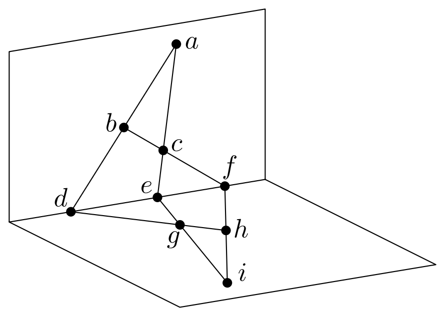
Total marks available: 52

When completing this assignment, you can use results stated in the course notes and lecture notes, but you should cite results that you use. In particular, you may use theorems from section 1 of the course notes that we have stated (but not yet proved). You should not use any other resources, printed or online. This is not a collaborative assignment – your submitted work should reflect your individual effort.

Q1. Consider the rank-4 matroid M with the geometric representation given below.



Give one example of each of the following:

(a) a minimum-sized circuit of M,

(b) a maximum-sized circuit of M,

(c) a basis of M,

(d) a minimum-sized independent set of M,

(e) a dependent set of M that is not a circuit,

(f) a set $X \subseteq E(M)$ with $r(X) = 3$ and $|X| = 5$. **[6]**

Q2. Let M be a matroid. We say that $x \in E(M)$ is a *loop* if $\{x\}$ is a circuit. Let $e \in E(M)$. Prove that the following are equivalent:

(i) e is a loop of M,

(ii) e is not in any basis of M,

(iii) e is not in any independent set of M. **[6]**

Q3. Let M_1 and M_2 be matroids on disjoint ground sets E_1 and E_2, respectively. Let $E = E_1 \cup E_2$ and $\mathcal{I} = \{I_1 \cup I_2 : I_1 \in \mathcal{I}(M_1) \text{ and } I_2 \in \mathcal{I}(M_2)\}$. Prove that there is a matroid M on ground set E whose family of independent set is $\mathcal{I}$. (Hint: you may use Theorem 1.7.)

[5]

Q4. Determine if the following statement is true or false: "If C and $(C - x) \cup y$ are both circuits in a matroid, where $x \in C$ and $y \notin C$, then $\{x, y\}$ is also a circuit." If true, prove it; if false, give a counterexample. **[3]**

Q5. Let $C_1, C_2, \ldots, C_k$ be pairwise disjoint circuits of a matroid M, where $k \geq 1$. Assume that M has a circuit not equal to any of $C_1, \ldots, C_k$. Let x_i be an element in C_i, for each i in $\{1, \ldots, k\}$. Prove that M has a circuit that does not contain any of $x_1, \ldots, x_k$. **[6]**

Q6. Recall that a *loop* in a matroid is a circuit of size one. A *parallel pair* in a matroid is a circuit of size two. A matroid is *simple* if it has no loops and no parallel pairs.

(a) How many non-isomorphic simple rank-3 matroids are there on six elements? Draw a geometric representation of each. **[6]**

(b) Let M be a matroid with rank 3. Prove that M is paving if and only if M is simple. **[3]**

Q7. Let E be a set, and let $\mathcal{I}$ be a family of subsets of E. For a set $Y \subseteq E$, when we say I is a *maximal subset of* Y *in* $\mathcal{I}$, we mean that $I \subseteq Y$ and $I \in \mathcal{I}$, and if $I' \in \mathcal{I}$ for some $I \subseteq I' \subseteq Y$, then $I = I'$.

(a) Let M be a matroid. Show that, for any subset X of $E(M)$, if I and I' are maximal subsets of X in $\mathcal{I}(M)$, then $|I| = |I'|$. **[3]**

(b) Suppose that $\mathcal{I}$ satisfies **I1** and **I2**, and, for any set $X \subseteq E$, if I and I' are maximal subsets of X in $\mathcal{I}$, then $|I| = |I'|$. Prove that $\mathcal{I}$ is the family of independent sets of a matroid with ground set E. **[5]**

Q8. Recall the following (see Exercise 1.4 or the exercise[1] at the end of lecture 2):

> Let E be a finite set, and let r be an integer such that $0 < r < |E|$. Let $\mathcal{C}'$ be a collection of r-element subsets of E such that if C_1 and C_2 are distinct members of $\mathcal{C}'$, then $|C_1 \cap C_2| < r - 1$. Let $\mathcal{B}$ be the family of r-element subsets of E that are not in $\mathcal{C}'$; that is, $\mathcal{B} = \{B \subseteq E : |B| = r \text{ and } B \notin \mathcal{C}'\}$. Then $(E, \mathcal{B})$ is a matroid.

We say that a matroid M is *sparse paving* if M is isomorphic to either $U_{0,n}$ or $U_{n,n}$ for some non-negative integer n, or we can choose some r and $\mathcal{C}'$ so that $M \cong (E, \mathcal{B})$.

(a) Prove that a sparse paving matroid is paving.

(b) Let $J(n, r)$ denote the simple graph that has r-element subsets of $\{1, 2, \ldots, n\}$ as its vertices, and two vertices are adjacent if and only if their intersection has cardinality $r - 1$. A *stable set* of a graph is a set of vertices that are pairwise non-adjacent. Draw $J(4, 2)$, and describe all stable sets of this graph.

(c) Describe all rank-2 sparse paving matroids on the ground set $\{1, 2, 3, 4\}$ (up to isomorphism) by providing the family of bases for each.

(d) Draw geometric representations of each of the matroids from (c). **[9]**

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[1] The exercise from the lecture was to prove this is indeed a matroid. For this assignment question you may assume this without proof.