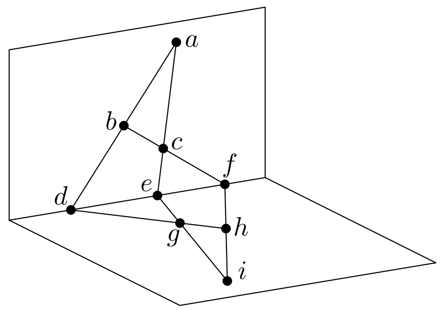
Total marks available: 52

Q1. Consider the rank-4 matroid M with the geometric representation given below.



Give one example of each of the following:

(a) a minimum-sized circuit of M,

Solution: The smallest circuits in this matroid have size three, so $\{a, b, d\}$ is a minimum-sized circuit (amongst several others).

(b) a maximum-sized circuit of M,

Solution: Since this matroid has rank 4, every set of size 5 is dependent, so circuits (being minimal dependent sets) have size at most 5 in this matroid. One example of a circuit with size 5 is $\{a, b, f, g, i\}$ (to see this, observe that any proper subset is independent, since any four such points are not coplanar).

(c) a basis of M,

Solution: Since this matroid has rank 4, we are looking for an independent set of size four. One example is $\{a, b, c, g\}$.

(d) a minimum-sized independent set of M,

Solution: The smallest independent set of M is the empty set $\emptyset$.

(e) a dependent set of M that is not a circuit,

Solution: Here we are looking for a dependent set that properly contains another dependent set. One example is $\{a, b, d, e\}$.

(f) a set $X \subseteq E(M)$ with $r(X) = 3$ and $|X| = 5$.

Solution: One solution here is $X = \{a, b, d, e, f\}$, or choose X to be any set of five points that are coplanar (and not collinear, but there are no 5 points on a line). **[6]**

Q2. Let M be a matroid. We say that $x \in E(M)$ is a *loop* if $\{x\}$ is a circuit. Let $e \in E(M)$. Prove that the following are equivalent:

(i) e is a loop of M,

(ii) e is not in any basis of M,

(iii) e is not in any independent set of M. **[6]**

Solution: Suppose that e is a loop. Then $\{e\}$ is a circuit, and in particular $\{e\}$ is a dependent set. Let I be an independent set of M. Then any subset of I is also independent, by (I2). In particular, $e \notin I$, for otherwise $\{e\}$ is independent. This shows that e is not in any independent set of M, i.e. (i) $\Rightarrow$ (iii).

Suppose that e is in some basis B of M. Then B is also an independent set of M (since a basis is a maximal independent set), so e is in an independent set of M. We've shown that the negation of (ii) implies the negation of (iii). Thus the contrapositive holds, that is, we've also shown that (iii) implies (ii).

Finally, suppose e is not in any basis of M. If $\{e\}$ is independent, then it is contained in some maximal independent set, contradicting that e is not in any basis. Therefore $\{e\}$ is dependent. By (I1), $\emptyset$ is independent, so $\{e\}$ is a minimal dependent set. Thus e is a loop of M. This shows that (ii) implies (i). It now follows that (i), (ii), and (iii) are equivalent, as required.

Q3. Let M_1 and M_2 be matroids on disjoint ground sets E_1 and E_2, respectively. Let $E = E_1 \cup E_2$ and $\mathcal{I} = \{I_1 \cup I_2 : I_1 \in \mathcal{I}(M_1) \text{ and } I_2 \in \mathcal{I}(M_2)\}$. Prove that there is a matroid M on ground set E whose family of independent set is $\mathcal{I}$. (Hint: you may use Theorem 1.7.)

[5]

Solution: By Theorem 1.7, it suffices to prove that $\mathcal{I}$ satisfies (I1), (I2), and (I3). Since M_1 and M_2 are matroids, we know (using Theorem 1.7 again) that the independent sets of these matroids $\mathcal{I}(M_1)$ and $\mathcal{I}(M_2)$ satisfy (I1), (I2), and (I3); we use this repeatedly below.

First we show that $\mathcal{I}$ satisfies (I1). Since $\emptyset$ is an independent set of both M_1 and M_2, we have that $\emptyset \cup \emptyset = \emptyset$ is in $\mathcal{I}$. So (I1) holds for $\mathcal{I}$.

Next, let $I \in \mathcal{I}$. Then $I = I_1 \cup I_2$ for some $I_1 \in \mathcal{I}(M_1)$ and $I_2 \in \mathcal{I}(M_2)$. For any $I' \subseteq I$, let $I'_1 = I' \cap I_1$ and $I'_2 = I' \cap I_2$. Then $I' = I'_1 \cup I'_2$, where $I'_1 \in \mathcal{I}(M_1)$ since $I'_1 \subseteq I_1$, and $I'_2 \in \mathcal{I}(M_2)$, since $I'_2 \subseteq I_2$. Hence $I' \in \mathcal{I}$, which shows that (I2) holds for $\mathcal{I}$.

Finally, let $I, I' \in \mathcal{I}$ where $|I| > |I'|$. We want to show (I3) holds, i.e. there exists an element $e \in I - I'$ such that $I' \cup e$ is in $\mathcal{I}$. Since $I, I' \in \mathcal{I}$, we have $I = I_1 \cup I_2$ and $I' = I'_1 \cup I'_2$ for some $I_1, I'_1 \in \mathcal{I}(M_1)$ and $I_2, I'_2 \in \mathcal{I}(M_2)$. As $|I| > |I'|$, either $|I_1| > |I'_1|$ or $|I_2| > |I'_2|$. Without loss of generality, we assume $|I_1| > |I'_1|$. Then, since (I3) holds for $\mathcal{I}(M_1)$, there exists an element $e \in I_1 - I'_1$ such that $I'_1 \cup e \in \mathcal{I}(M_1)$. Now $(I'_1 \cup e) \cup I'_2$ is in $\mathcal{I}$, which shows that (I3) does indeed hold.

Q4. Determine if the following statement is true or false: "If C and $(C - x) \cup y$ are both circuits in a matroid, where $x \in C$ and $y \notin C$, then $\{x, y\}$ is also a circuit." If true, prove it; if false, give a counterexample. **[3]**

Solution: This is false: we give a counterexample. Consider the matroid on the ground set $\{x, y, z, w\}$ that is isomorphic to $U_{2,4}$ (so a subset of $\{x, y, z, w\}$ is independent if and only if it has cardinality at most two). Let C be $\{x, z, w\}$. Then C and $(C - x) \cup y$ are both circuits, but $\{x, y\}$ is not.

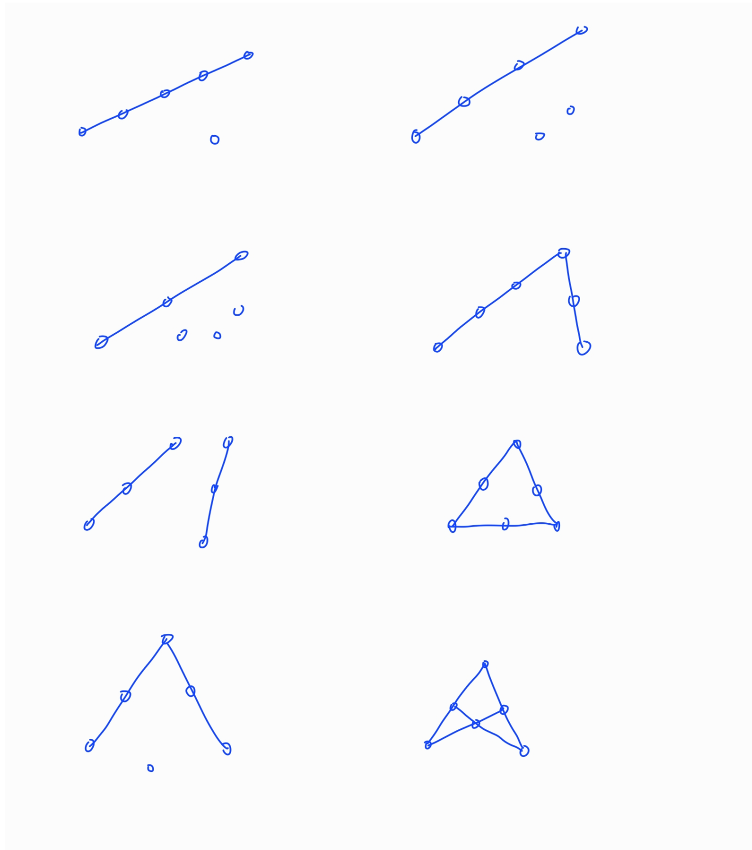
Q5. Let $C_1, C_2, \ldots, C_k$ be pairwise disjoint circuits of a matroid M, where $k \geq 1$. Assume that M has a circuit not equal to any of $C_1, \ldots, C_k$. Let x_i be an element in C_i, for each i in $\{1, \ldots, k\}$. Prove that M has a circuit that does not contain any of $x_1, \ldots, x_k$. **[6]**

Solution: Consider all circuits that are not in the collection $\{C_1, \ldots, C_k\}$. There is at least one such circuit, by hypothesis. Amongst all such circuits, let us choose C so that $|C \cap \{x_1, \ldots, x_k\}|$ is as small as possible. If C does not contain any of $x_1, \ldots, x_k$ then C is the circuit we desire. Therefore let us assume that C contains x_i for some i. Now $C \neq C_i$, by our choice of C. Since x_i is in $C \cap C_i$, we apply circuit exchange, and we find a circuit C' contained in $(C \cup C_i) - x_i$. We ask if C' could be equal to one of the circuits C_j. If so, then $j \neq i$, since C' does not contain x_i. So assume that $C' = C_j$. But any element of C_j is not contained in C_i, since $C_i \cap C_j = \emptyset$. This means that every element of C_j is contained in C, since any such element is contained in $(C \cup C_i) - x_i$. This means that $C_j \subseteq C$, implying $C_j = C$. This is a contradiction as C was assumed to be not equal to any circuit in $\{C_1, \ldots, C_k\}$. Now we know that C' is also not equal to any circuit in $\{C_1, \ldots, C_k\}$. If C' contains an element $x_j \neq x_i$, then this element was in C, since x_j is not in C_i because $C_i \cap C_j = \emptyset$. This shows that $|C' \cap \{x_1, \ldots, x_k\}| < |C \cap \{x_1, \ldots, x_k\}|$, contradicting our choice of C. Therefore C contains no elements of $\{x_1, \ldots, x_k\}$ and we are done.

Q6. Recall that a *loop* in a matroid is a circuit of size one. A *parallel pair* in a matroid is a circuit of size two. A matroid is *simple* if it has no loops and no parallel pairs.

(a) How many non-isomorphic simple rank-3 matroids are there on six elements? Draw a geometric representation of each. **[6]**

Solution: There are 9. One is the uniform matroid $U_{3,6}$, but there are 8 others, as drawn below:



(b) Let M be a matroid with rank 3. Prove that M is paving if and only if M is simple. **[3]**

Solution: Suppose that M is paving. Then, since $r(M) = 3$, the circuits of M have size at least three. Thus M has no circuits of size one or two, that is, M has no loops and no parallel pairs.

For the converse, suppose that M is simple. Then M has no loops or parallel pairs. That is, M has no circuits of size one or size two. Since the empty set $\emptyset$ is independent (as the independent sets of M satisfy (I1) by Theorem 1.7), M also has no circuits of size zero. Thus any circuit of M has size at least three. Hence, as $r(M) = 3$, the matroid M is paving.

Q7. Let E be a set, and let $\mathcal{I}$ be a family of subsets of E. For a set $Y \subseteq E$, when we say I is a *maximal subset of* Y *in* $\mathcal{I}$, we mean that $I \subseteq Y$ and $I \in \mathcal{I}$, and if $I' \in \mathcal{I}$ for some $I \subseteq I' \subseteq Y$, then $I = I'$.

(a) Let M be a matroid. Show that, for any subset X of $E(M)$, if I and I' are maximal subsets of X in $\mathcal{I}(M)$, then $|I| = |I'|$. **[3]**

(b) Suppose that $\mathcal{I}$ satisfies **I1** and **I2**, and, for any set $X \subseteq E$, if I and I' are maximal subsets of X in $\mathcal{I}$, then $|I| = |I'|$. Prove that $\mathcal{I}$ is the family of independent sets of a matroid with ground set E. **[5]**

Solution:

(a) Assume that I and I' are maximal subsets of X in $\mathcal{I}(M)$, but that $|I| \neq |I'|$. Without loss of generality, we can assume that $|I| < |I'|$. Then **I3** implies that there is an element, $e \in I' - I$, such that $I \cup e$ is in $\mathcal{I}(M)$. But $I' \subseteq X$ implies that $e \in X$, and hence $I \cup e \subseteq X$. Moreover, I is a proper subset of $I \cup e$. This contradicts the fact that I is a maximal subset of X in $\mathcal{I}(M)$.

(b) Since $\mathcal{I}$ satisfies **I1** and **I2**, it suffices to show that **I3** holds. Let I_1 and I_2 be members of $\mathcal{I}$ where $|I_2| < |I_1|$. Let $X = I_1 \cup I_2$. Since $I_1 \in \mathcal{I}$ and $I_1 \subseteq X$, there is a maximal subset of X in $\mathcal{I}$ that contains I_1. Let this maximal subset be I_1'. Similarly, let I_2' be a maximal subset of X in $\mathcal{I}$ that contains I_2. Then $|I_1'| = |I_2'|$ by hypothesis, so $|I_2| < |I_1| \leq |I_1'| = |I_2'|$. Therefore there is an element $e \in I_2' - I_2$. Now $I_2 \cup e \subseteq I_2'$, so $I_2 \cup e \in \mathcal{I}$ by **I2**. Also, $e \in X - I_2$, so $e \in I_1 - I_2$. Therefore **I3** holds.

Q8. Recall the following (see Exercise 1.4 or the exercise[1] at the end of lecture 2):

> Let E be a finite set, and let r be an integer such that $0 < r < |E|$. Let $\mathcal{C}'$ be a collection of r-element subsets of E such that if C_1 and C_2 are distinct members of $\mathcal{C}'$, then $|C_1 \cap C_2| < r - 1$. Let $\mathcal{B}$ be the family of r-element subsets of E that are not in $\mathcal{C}'$; that is, $\mathcal{B} = \{B \subseteq E : |B| = r \text{ and } B \notin \mathcal{C}'\}$. Then $(E, \mathcal{B})$ is a matroid.

We say that a matroid M is *sparse paving* if M is isomorphic to either $U_{0,n}$ or $U_{n,n}$ for some non-negative integer n, or we can choose some r and $\mathcal{C}'$ so that $M \cong (E, \mathcal{B})$.

(a) Prove that a sparse paving matroid is paving.

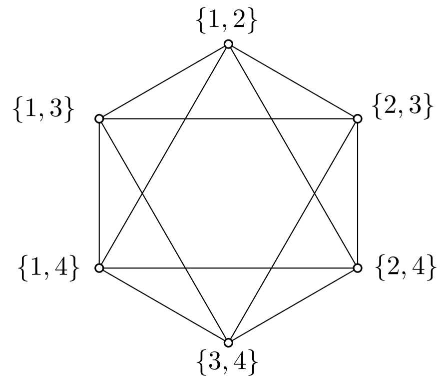
Solution: Let M be sparse paving. If M is uniform, then the circuits have size at least $r+1$ (in fact, precisely $r+1$), so M is paving. So we may assume that M is not uniform. Then there exists some integer r such that $0 < r < |E|$, and some collection $\mathcal{C}'$ of r-element subsets of E such that if C_1 and C_2 are distinct members of $\mathcal{C}'$, then $|C_1 \cap C_2| < r - 1$, and $M \cong (E, \mathcal{B})$ where $\mathcal{B} = \{B \subseteq E : |B| = r \text{ and } B \notin \mathcal{C}'\}$.

Towards a contradiction, suppose there is a circuit C of M with $|C| < r$. Then $|C| \leq r - 1$, so we can choose an $(r-1)$-element set C' such that $C \subseteq C' \subseteq E$. Now C' is a dependent set of M of size $r - 1$. Moreover, as $r < |E|$, we have $r \leq |E| - 1$, so $|C'| = r - 1 \leq |E| - 2$. Thus there are distinct elements $x, y \in E - C'$, so that $C' \cup \{x\}$ and $C' \cup \{y\}$ are distinct r-element sets. As $C' \cup \{x\}$ and $C' \cup \{y\}$ contain C, they are dependent. Since they are r-element subsets of E that are not bases, $C' \cup \{x\}$ and $C' \cup \{y\}$ are members of $\mathcal{C}'$. But $|(C' \cup \{x\}) \cap (C' \cup \{y\})| = |C'| = r - 1$, a contradiction. Thus every circuit of M has size at least r, so M is paving.

(b) Let $J(n, r)$ denote the simple graph that has r-element subsets of $\{1, 2, \ldots, n\}$ as its vertices, and two vertices are adjacent if and only if their intersection has cardinality $r - 1$. A *stable set* of a graph is a set of vertices that are pairwise non-adjacent. Draw $J(4, 2)$, and describe all stable sets of this graph.

Solution:

[1] The exercise from the lecture was to prove this is indeed a matroid, but for this assignment question you may assume this without proof.



The empty set is a stable set, any singleton consisting of a single vertex of $J(4,2)$ is a stable set, and there are three stable sets consisting of two vertices: $\{\{1,2\},\{3,4\}\}$, $\{\{1,4\},\{2,3\}\}$, and $\{\{1,3\},\{2,4\}\}$.

(c) Describe all rank-2 sparse paving matroids on the ground set $\{1,2,3,4\}$ (up to isomorphism) by providing the family of bases for each.

Solution: By (b), up to symmetry there are three different stable sets to consider. Thus, up to isomorphism, the only rank-2 sparse paving matroids on $\{1,2,3,4\}$ have the following families of bases:

$$\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$$

$$\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$$

$$\{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$$

(d) Draw geometric representations of each of the matroids from (c). **[9]**

Solution: