MATH432 | Lecture 9

Assignment 2 due Mon 15 April

Last time: duality and representable matroids

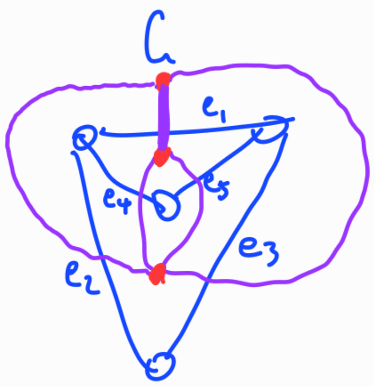
Today: duality and graphic matroids

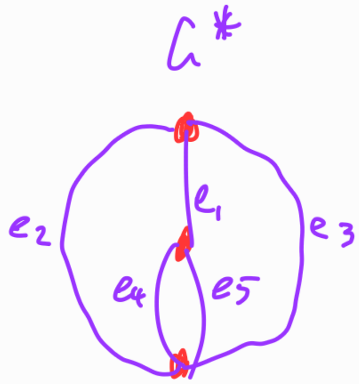
First, we recall planar graphs and their duals.

A plane graph is a graph together with an embedding in the plane (mapping each vertex to a point, each edge to a simple curve between the points that are the image of the ends of the edge, and the only points where curves meet is at the image of a vertex).

A graph is planar if it has a planar embedding.

Any plane graph G has a geometric dual G^*.





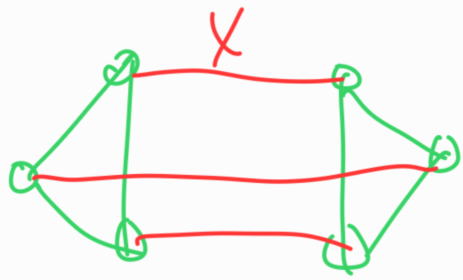
Let $G = (V, E)$ be a graph, and $X \subseteq E$.

Then $G \setminus X$ denotes the subgraph $(V, E - X)$.

The set X is an edge cut in G if $G \setminus X$ has more components than G.

A **bond** is a minimal edge cut.

e.g.



X is a minimal edge cut.

i.e. X is a bond.

Thm Let G be a connected plane graph.

i) If C is a cycle of G, then C is a bond in G^*.

ii) If C is a bond of G, then C is a cycle in G^*.

Proposition 3.19: Let G be a graph with edge set E and let $X \subseteq E$. Then

X is coindependent in $M(G)$ if and only if X is not an edge cut in G.

Corollary 3.20: Let G be a graph and $X \subseteq E(G)$.

X is a cocircuit $\iff$ X is a bond of G.

Proof of proposition 3.19:

$(\Rightarrow)$ Suppose X is an edge cut of G.

Let k be the number of components of G.

Then $G \setminus X$ has at least $k+1$ components.

Now the size of a maximal forest in G is $|V(G)| - k$, whereas the size of a maximal forest in $G \setminus X$ is at most

$|V(G)| - (k+1)$ (by Corollary 2.12).

That is, the size of a maximal forest of $G \setminus X$ is smaller than the size of a maximal forest of G.

This implies that no maximal forest of G is contained in $E-X$, so there is no basis of $M(G)$ contained in $E-X$.

Since $E-X$ is not spanning, X is not codependent.

(by Proposition 3.4).

($\Leftarrow$) Now suppose X is not an edge cut in G.

Then the size of a maximal forest in G is the same as the size of a maximal forest in $G \setminus X$

(by Corollary 2.12).

In particular, there is some maximal forest of G contained in $E-X$, which means there is some basis of $M(G)$ contained in $E-X$. So $E-X$ is spanning and hence (by Prop 3.4) X is coindependent. $\square$

For a graphic matroid $M(G)$, we denote the dual as $M^*(G)$, i.e. $M^*(G) = (M(G))^*$.

<u>Theorem</u> 3.26 Let G be a planar graph. Then $M^*(G)$ is graphic. Moreover, if G is a plane graph with geometric dual G^*, then

$$M^*(G) = M(G^*).$$

<u>Proof:</u> It suffices to show that the cycles of G^* are precisely the bonds of G, because then

$\mathcal{C}$ is the family of cycles of G^*

$\iff$ $\mathcal{C}$ is the family of bonds of G

we'll show this

$\iff$ $\mathcal{C}$ is the family of cocircuits of $M(G)$.

by Corollary 3.20

i.e the circuits of $M(G^*)$ are precisely the circuits of $M^*(G)$.

so $M(G^*) = M^*(G)$.

1)

First consider a cycle C in G^*.

Plan:

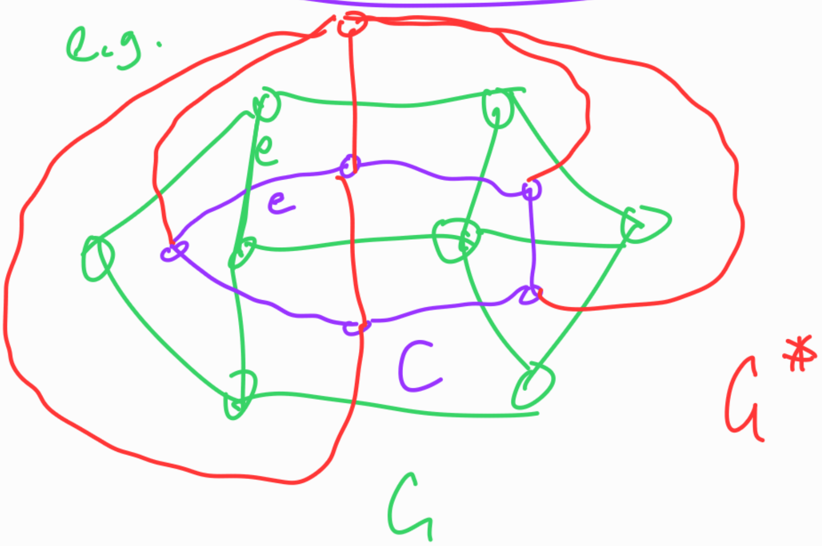
1) cycle in G^* contains a bond of G

2) bond of G contains a cycle of G^*

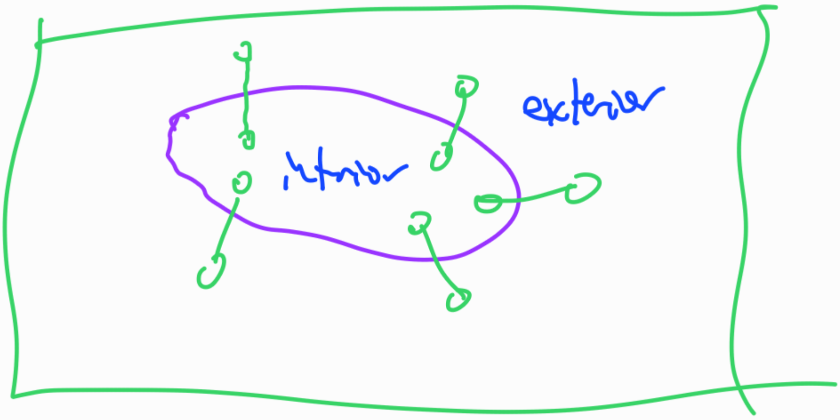
C is a cycle of faces in G

There is a simple curve that traverses these faces in the planar embedding.

e.g.



By the Jordan Curve Theorem, the curve divides the plane into two regions: an interior and an exterior.

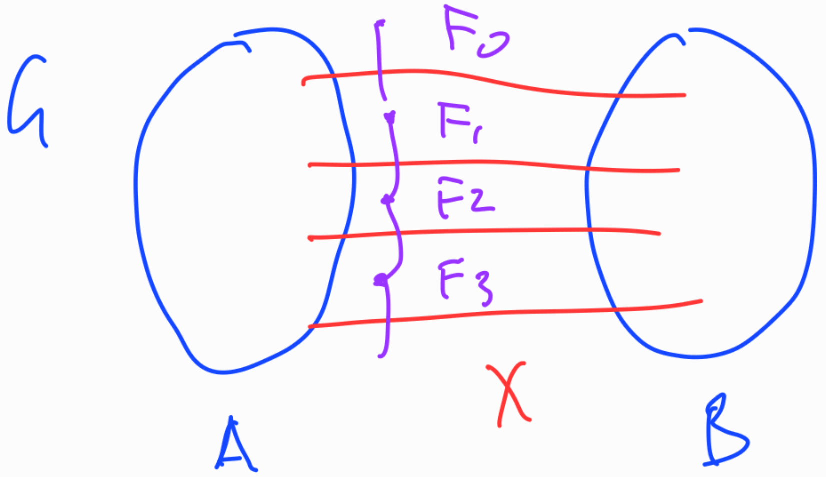


Consider each edge e that joins a vertex in the interior to the exterior. These edges correspond to the cycle C.

Since $|C| \geq 1$, there is at least one vertex in the interior and at least one vertex in the exterior. It follows that C is an edge cut of G, so C contains a bond.

2) Now let X be a bond in G.

Then there is a partition (A, B) of $V(G)$ such that each edge in X joins a vertex in A to a vertex in B, and there is no path in $G\setminus X$ from a vertex in A to a vertex in B.

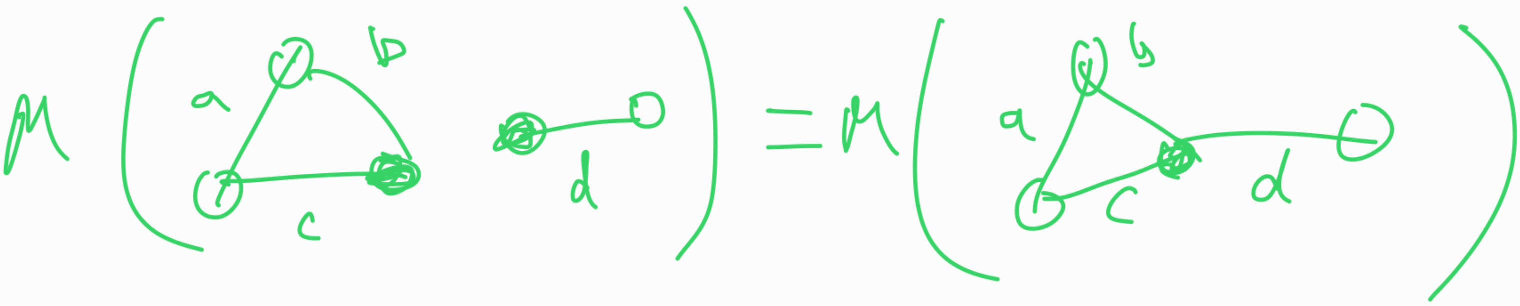


Let $\mathcal{F}$ be the set of faces of G that are incident to at least one edge of X. If $|\mathcal{F}|=1$, then $|X|=1$ and the singleton X corresponds to a loop in G^*, so is a cycle. So suppose $|\mathcal{F}| \geq 2$.

Now each $F \in \mathcal{F}$ is bounded by a cycle in G, where at least two edges of this cycle are in X, for otherwise $G\setminus X$ has a path from a vertex in A to a vertex in B. Thus, in the dual, each vertex corresponding to $F \in \mathcal{F}$ has at least two incident edges in X. A graph where every vertex has degree at least 2 must contain a cycle. So there is a cycle in G^* whose edges are contained in X. This proves that the cycles of G^* are precisely the bonds of G, as desired. $\square$

Is a graphic matroid $M(G)$ uniquely determined by the graph G? No.

e.g.



Proposition 3.16: Suppose M is a graphic matroid. Then $M = M(G)$ for some connected graph.