MATH432

Assignment 1 due Friday 5pm

Recall: Let M be a rank-r matroid.

- All circuits have size at most $r+1$
- M is uniform if all circuits have size $r+1$
- M is paving if all circuits have size r or $r+1$.

Recap: last time we saw matroids that arise from a set of vectors in $\mathbb{F}^m$ (or, a matrix over $\mathbb{F}$) – these are called $\mathbb{F}$-representable matroids

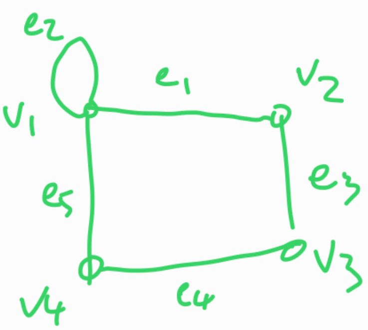
Today, matroids that arise from graphs.

Recall: a graph consists of $G=(V,E)$

- a set V called the vertex set
- a set E called the edge set

and an incidence function φ that maps each $e \in E$ to either a pair in V, or a singleton in V

e.g.



$V=\{v_1,v_2,v_3,v_4\}$

$E=\{e_1,\dots,e_5\}$

$\varphi(e_1)=\{v_1,v_2\}$, $\varphi(e_2)=\{v_1\}$, ...

adjacent edges

when a vertex and edge are incident

ends of edge

a walk in a graph

path

closed walk — first and last vertex are the same.

cycle

a subgraph of $G = (V, E)$ is a graph $G' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$ and $\varphi(e') \subseteq V'$ for all $e' \in E'$.

a subgraph G' of $G = (V, E)$ is edge-induced, for some $E' \subseteq E$ when $G' = (V', E')$ with $V' = \bigcup_{e \in E'} \varphi(e)$. We denote G' as $G[E']$.

Theorem: Let G be a graph with edge set E.

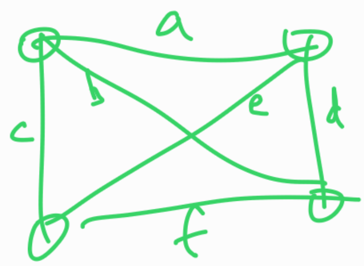
Let $\mathcal{C}$ be

$\{ C \subseteq E : C$ is the edge set of a cycle in $G \}$.

Then $\mathcal{C}$ is the family of circuits of a matroid on ground set E.

We denote this matroid as $M(G)$.

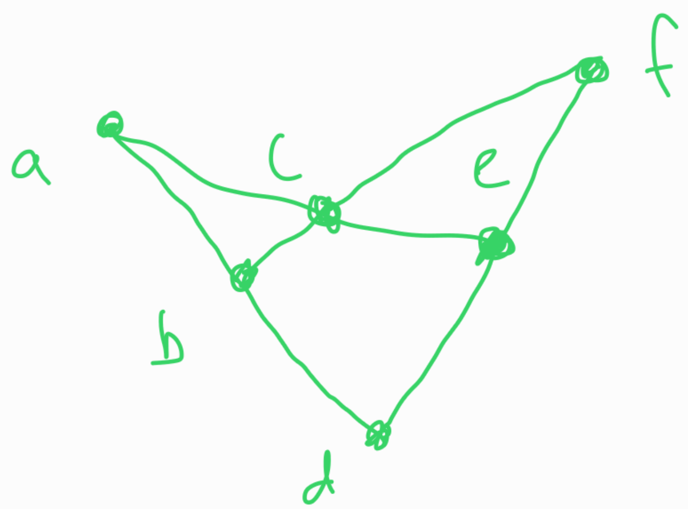
e.g. Consider the graph K_4 on edge set $E = \{a,b,c,d,e,f\}$ as illustrated.



$$\mathcal{C} = \{\{a,b,d\}, \{a,c,e\}, \{d,e,f\}, \{b,c,f\}, \{a,c,d,f\}, \{a,b,e,f\}, \{b,c,d,e\}\}$$

The bases have size 3, so $M(K_4)$ has rank 3.

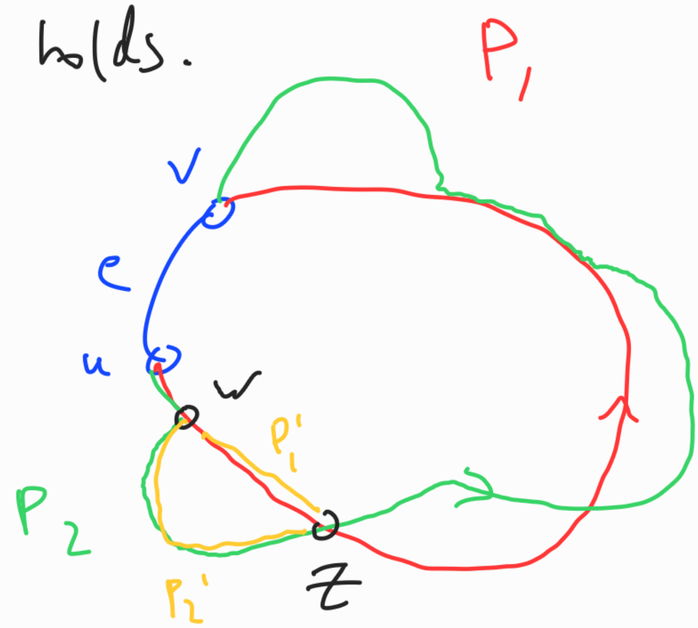
A geometric representation for $M(K_4)$:



Proof of Thm: We will show $\mathcal{C}$ satisfies (C1)–(C3).

Since a cycle of a graph contains at least one edge, $\emptyset \notin \mathcal{C}$, so (C1) holds. Moreover, no cycle in a graph contains a smaller cycle as a proper subgraph, so (C2) holds.

It remains to show (C3) holds.



Let C_1 and C_2 be edge sets of distinct cycles in G, with $e \in C_1 \cap C_2$. We will show that there is a cycle on edge set C_3 where $C_3 \subseteq (C_1 \cup C_2) - \{e\}$.

Note that e is not a loop, otherwise $C_1 = C_2 = \{e\}$.

Let u and v be the ends of e. Let P_i be the path from u to v along the edges in $C_i - \{e\}$, for $i \in \{1, 2\}$. Let $P_1 = v_0, e_1, v_1, \dots, v_k$

and $P_2 = v_0', e_1', v_1', \dots, v_{k'}'$

where $v_0 = v_0' = u$ and $v_k = v_{k'}' = v$.

Let w be the first vertex v_i in P_1 for which e_{i+1} is not in P_2.

Traverse P_1, starting at w, towards v, and let z be the first vertex distinct from w that is in P_2 (such a vertex exists since v is one such vertex). Call this path P_1'. Now let P_2' be the subpath of P_2 obtained by starting at w and ending at z. Now P_1' and P_2' are both paths from w to z, with no other vertices in common, so letting C_3 be the union of the edge sets of P_1' and P_2', C_3 is the edge set of a cycle contained in $(C_1 \cup C_2) - \{e\}$.

Thus (C3) holds. $\square$

For a graph G, we call this matroid $M(G)$ the cycle matroid of G.

A matroid M is graphic if there exists a graph G such that $M \cong M(G)$.

For a graph G, a forest of G is a set $F \subseteq E(G)$ such that $G[F]$ contains no cycles, and a forest F is maximal if there is no proper superset $F' \subseteq E(G)$ of F such that F' is a forest of G.

For a graph G

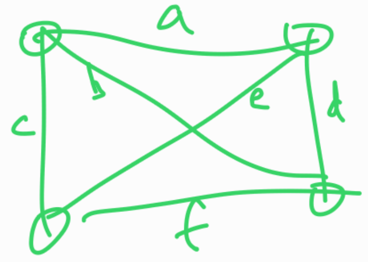
$$\left.\begin{array}{l}\text{circuits in } M(G)\\ \text{independent sets}\\ \text{bases}\end{array}\right\} \text{ correspond to } \left\{\begin{array}{l}\text{cycles in } G\\ \text{forests of } G\\ \text{maximal forests of } G.\end{array}\right.$$

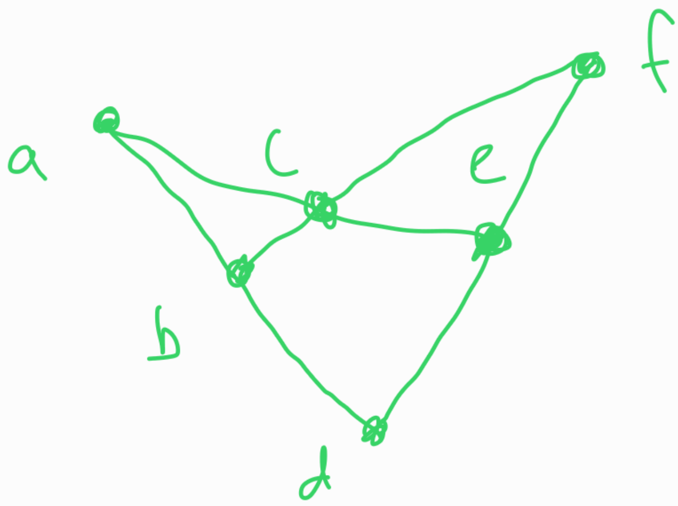
respectively.

When G is connected, the bases of $M(G)$ are the edge sets of spanning trees of G.

A spanning tree of a graph G is a subgraph G' of G such that G' is a tree and $V(G) = V(G')$

e.g. revisited: Consider the graph K_4 on edge set $E = \{a,b,c,d,e,f\}$ as illustrated.





$$A = \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

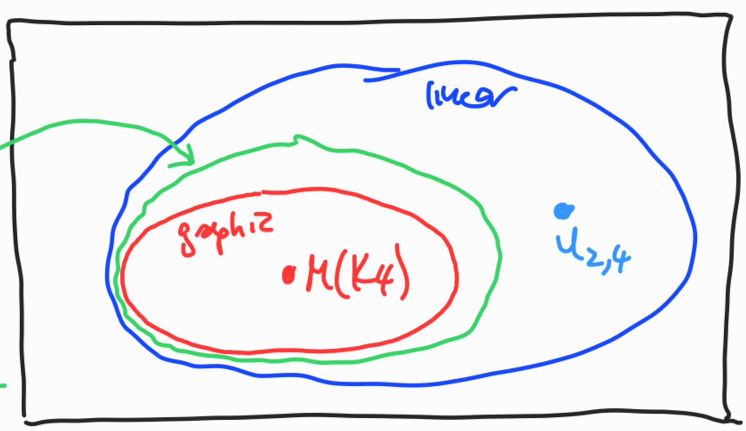
over $GF(2)$

$$M(K_4) \cong M[A]$$

This demonstrates that $M(K_4)$ is $GF(2)$-representable.

In fact, we will see for any graph G, the matroid $M(G)$ is $\mathbb{F}$-representable for every field $\mathbb{F}$.

regular matroids: matroids representable over every field.



Exercise: Prove that $U_{2,4}$ is not graphic and is not $\mathbb{F}$-representable for each field $\mathbb{F}$.

However: when $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ over $\mathbb{R}$

then $M[A] \cong U_{2,4}$, so $U_{2,4}$ is $\mathbb{R}$-representable

(and is therefore linear).

Exercise: (1) Characterise for which fields $\mathbb{F}$ the matroid $U_{2,4}$ is $\mathbb{F}$-representable.

(2) What about $U_{2,n}$ (for any $n \geq 2$).

Note: in general, characterising which fields a uniform matroid $U_{r,n}$ is representable over is a hard problem; no complete answer is known when $r \geq 5$.