MATH4432 | lecture 3

Recap: · defined a matroid $(E, \mathcal{B})$ in terms of bases $\mathcal{B}$

($\mathcal{B}$ satisfies (B1) and (B2))

· seen an equivalent definition using independent sets $\mathcal{I}$

see Thm 1.7 ($\mathcal{I}$ satisfies (I1)–(I3))

→ a basis is a maximal independent set

· seen an equivalent definition using circuits $\mathcal{C}$

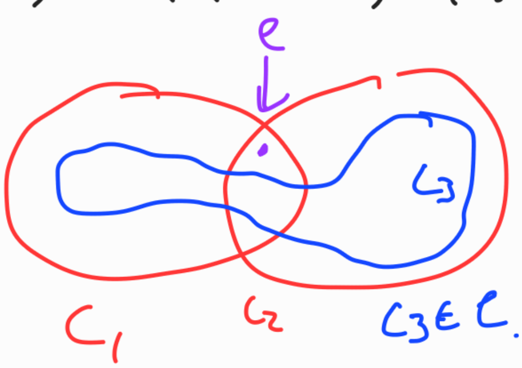
see Thm 1.11 ($\mathcal{C}$ satisfies (C1)–(C3))

→ a circuit is a minimal dependent set

(C1) $\emptyset \notin \mathcal{C}$.

(C2) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C3) If C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - \{e\}$.



Prop 1.12 Let M be a matroid, let I be an independent set of M, and $e \in E(M) - I$ such that $I \cup e$ is dependent. Then $I \cup e$ contains a unique circuit C, and $e \in C$.

Proof: Certainly $I \cup e$ contains a circuit, since $I \cup e$ is dependent.

Moreover, for any circuit C contained in $I \cup e$, if $e \notin C$, then I contains the circuit C, which is contradictory (by I2). So for any circuit $C \subseteq I \cup e$, we have $e \in C$. It remains only to show that there is a unique circuit contained in $I \cup e$. Suppose there are distinct circuits $C, C' \subseteq I \cup e$. Then $e \in C \cap C'$. Then, by (C3) there exists a circuit

$$C'' \subseteq (C_1 \cup C_2) - \{e\},$$ which is a contradiction.

So there is a unique circuit C contained in $I \cup e$. □

Let M be a matroid. Suppose B is a basis, and $e \in E(M) - B$. Then, by Proposition 1.12, there is a unique circuit contained in $B \cup e$, which we call the fundamental circuit of e with respect to B, and denote it $C(e, B)$.

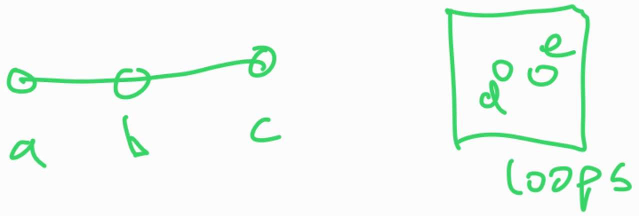
Def^n: a circuit of size 1 is called a loop.

e.g. In $U_{0,1}$, with $E(U_{0,1}) = \{a\}$

a is a loop.

Note: geometrically, we denote the loops in a box to the side

e.g.



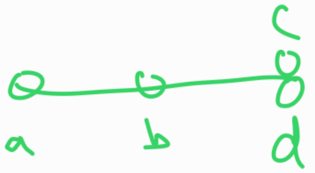
Def^n: when a pair $\{a,b\}$ is a circuit in a matroid, we call $\{a,b\}$ a parallel pair.

Geometrically, a parallel pair is a pair of coincident points.

Recall the rank of a matroid M is the size of a basis of M

and the rank of a set $X \subseteq E(M)$ is the size of a maximal independent set contained in X.

e.g. 1 Consider the rank-2 matroid with geometric representation:



$$r(\{a,b,d\}) = 2 = r(\{a,b,c,d,e\})$$

$$r(\{c,d,e\}) = 1$$

$$r(\{e\}) = 0 \qquad r(\{a\}) = 1$$

Let M be a matroid and $X \subseteq E(M)$

Then

$$r(X) = |X| \iff X \in \mathcal{I}(M)$$

If $C \in \mathcal{C}(M)$, then $r(C) = |C| - 1$.

Note, the converse does not hold.

e.g.

$$r(\{a,b,c\}) = 2$$

$$= |\{a,b,c\}| - 1$$

but $\{a,b,c\}$ is not a circuit.

Let E be a set and let $r : 2^E \to \mathbb{N}$ be a function.

Consider the following properties of r:

(R1) $0 \leq r(X) \leq |X|$ for all $X \subseteq E$

(R2) $r(Y) \leq r(X)$ for all $Y \subseteq X \subseteq E$

(R3) $r(X \cap Y) + r(X \cup Y) \leq r(X) + r(Y)$ for all $X, Y \subseteq E$

(R3) is called submodularity.

Thm 1.16: Let M be a matroid with rank function r. Then r satisfies (R1)-(R3).

Conversely, let E be a set and let r be a function $r : 2^E \to \mathbb{N}$ that satisfies (R1)-(R3). Then r is the rank function of a matroid M with $\mathcal{I}(M) = \{X \subseteq E : r(X) = |X|\}$

The proof is deferred for now.

We return to the proof of Thm 1.7.

Recall we've seen one direction ("Lemma A").

<u>Lemma B</u>: Let E be a set and let $\mathcal{I}$ be a family of subsets of E satisfying (I1)–(I3). Let $\mathcal{B}$ consist of the maximal members of $\mathcal{I}$. Then $\mathcal{B}$ satisfies (B1) and (B2).

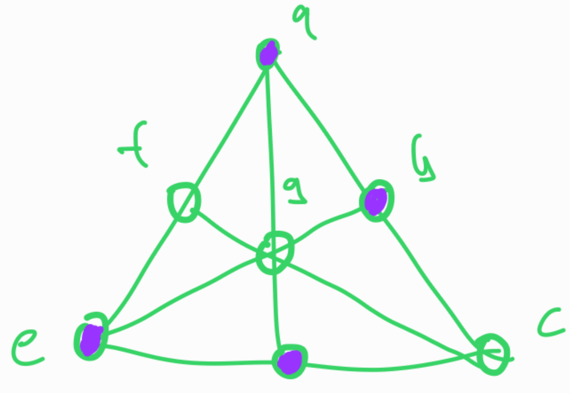
<u>Proof:</u> By (I1), $\emptyset \in \mathcal{I}$, so there is a maximal member of $\mathcal{I}$ (which contains $\emptyset$), which is in $\mathcal{B}$ (by def^n of $\mathcal{B}$). So $\mathcal{B} \neq \emptyset$, satisfying (B1).

Next we show (B2) holds. First we claim each member of $\mathcal{B}$ has the same size. Suppose not. Then there exists $B_1, B_2 \in \mathcal{B}$ such that $|B_1| > |B_2|$. Then, as $B_1, B_2 \in \mathcal{I}$ by (I3) there exists $e \in B_1 - B_2$ such that

$B_2 \cup \{e\} \in \mathcal{I}$, which contradicts that B_2 is maximal. So all members of $\mathcal{B}$ have the same size.

Now suppose $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$. Then $|B_1| = |B_2|$, so $|B_1 - \{x\}| < |B_2|$. Now $B_1 - \{x\} \in \mathcal{I}$ and so by (I3) there exists $y \in B_2 - (B_1 - \{x\})$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{I}$. Note $y \neq x$ since $x \notin B_2$, so $y \in B_2 - B_1$. Since $|(B_1 - \{x\}) \cup \{y\}| = |B_2|$ it follows from the claim that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$ so (B2) holds. $\square$

<u>example</u> Consider the rank-3 matroid with the following geometric representation



This configuration of points is known as the Fano plane

The matroid is called the

d Fano matroid, denoted F_7.

$$\{\{a,b,d\},\{a,d,f\}\} \subseteq \mathcal{B}(F_7)$$

$$\{\{a,b,c\},\{b,d,f\},\{a,b,d,e\}\} \in \mathcal{C}(F_7)$$