MATH432 | Lecture 18

Assignment 4 due 17 May

Test 10am 27 May

Last time:

For a matroid M, and $X \subseteq E(M)$, we defined the connectivity of X:

$$\lambda(X) = r(X) + r(E-X) - r(M)$$

Note: $\lambda(E(M)-X) = \lambda(X)$. We say λ is symmetric.

A partition $(X, E(M)-X)$ is a j-separation if

$|X| \geq j$ and $|E(M)-X| \geq j$ and $\lambda(X) < j$

(Note $\lambda(E(M)-X) < j$)

We say M is k-connected if M has no j-separations for $j < k$.

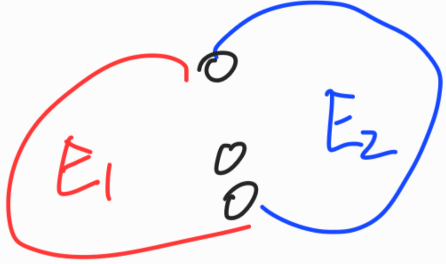
A partition $(X, E(M)-X)$ is a vertical j-separation if

$r(X) \geq j$ and $r(E(M)-X) \geq j$ and $\lambda(X) < j$

We say M is vertically k-connected if M has no vertical j-separations for $j < k$.

Matroid connectivity vs graph connectivity

Consider a connected graph G with a k-separation (E_1, E_2) and the cycle matroid $M(G)$.



We have $|V(G(E_1))| > k$ and $|V(G(E_2))| > k$

and $|V(G[E_1]) \cap V(G[E_2])| \leq k$.

Assuming $G[E_1]$ and $G[E_2]$ are connected,

$$r_{M(G)}(E_i) = |V(G[E_i])| - 1$$

So $r_{M(G)}(E_i) \geq k$.

$$|V(G[E_1]) \cap V(G[E_2])| = |V(G[E_1])| + |V(G[E_2])| - |V(G)|$$

$$= r_{M(G)}(E_1) + 1 + r_{M(G)}(E_2) + 1 - (r(M(G)) + 1)$$

$$= r_{M(G)}(\grave{E}_1) + r_{M(G)}(E_2) - r(M(G)) + 1$$

$$= \lambda_{M(G)}(\grave{E}_1) + 1 .$$

So $\lambda_{M(G)}(\grave{E}_1) < k$.

That is, $(\grave{E}_1, E_2)$ is a vertical k-separation in $M(G)$.

The connectivity function is invariant under duality:

Prop 8.18: $\lambda_M(X) = \lambda_{M^*}(X)$ for all $X \subseteq E(M)$.

Let $E = E(M)$.

Lemma 8.17 $\lambda_M(X) = r(X) + r^*(X) - |X|$.

Proof: Recall $r^*(X) = |X| + r(E-X) - r(M)$. for any $X \subseteq E$ (see Prop 3.9).

Thus $\lambda(X) = r(X) + r(E-X) - r(M)$

$$= r(X) + \left(r^*(X) - |X| + \cancel{r(M)} - \cancel{r(M)}\right)$$

$$= r(X) + r^*(X) - |X|$$

as required. □

Proof of Prop 8.18: From Lemma 8.17:

$$\lambda_M(X) = r_M(X) + r^*(X) - |X|$$
$$= r_{M^*}(X) + r^*_{M^*}(X) - |X| = \lambda_{M^*}(X)$$

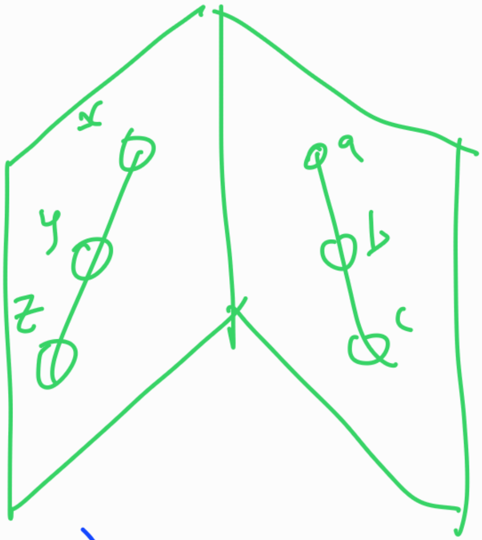
as required. □

It follows from Prop 8.18 that

M is k-connected $\iff$ M^* is k-connected.

However, the same is not true of vertical k-connectivity.

Example 1 Consider the rank-4 matroid M, illustrated here:



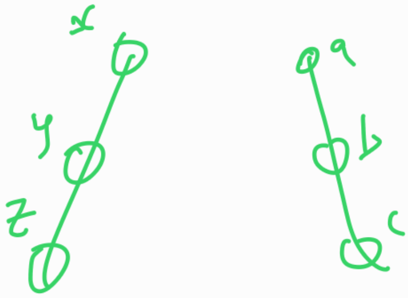
Consider $(\{x,y,z\},\{a,b,c\})$

$$\lambda_M(\{x,y,z\}) = r(\{x,y,z\}) + r(\{a,b,c\}) - r(M)$$

$$= 2+2-4 = 0 < 1$$

So $(\{x,y,z\},\{a,b,c\})$ is a $\underline{1}$-separation and hence M_1 is not 2-connected.

<u>Example 2</u> Consider the rank-3 matroid M_2, illustrated here:



Consider $(\{x,y,z\},\{a,b,c\})$

$$\lambda_M(\{x,y,z\}) = 2+2-3$$

$$= 1 < 2$$

So $(\{x,y,z\},\{a,b,c\})$ is a $\underline{2}$-separation and hence M_2 is not 3-connected.

It is 2-connected however (you can check that is has no 1-separations).

1-separations in fact correspond to a natural operation on matroids

Let M_1 and M_2 be matroids on disjoint ground sets. We define the direct sum of M_1 and M_2, denoted $M_1 \oplus M_2$, to be the matroid on ground set $E(M_1) \cup E(M_2)$ with independent sets

$$\{ I_1 \cup I_2 : I_1 \in \mathcal{I}(M_1) \text{ and } I_2 \in \mathcal{I}(M_2) \}$$

We proved this is a matroid in an assignment Q

Prop 8.3

$$r(M_1 \oplus M_2) = r(M_1) + r(M_2) = r_{M_1 \oplus M_2}(E_1) + r_{M_1 \oplus M_2}(E_2)$$

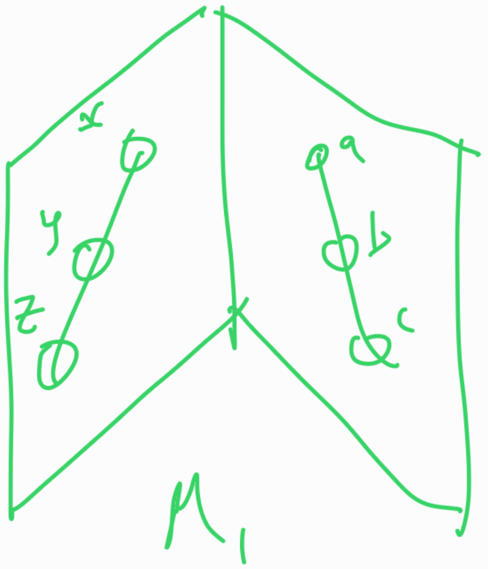
Prop 8.4: Let M be a matroid with a 1-separation (X, Y). Then $M = (M|X) \oplus (M|Y)$

$M|X = M \setminus (E-X)$

Since a matroid is 2-connected if it has no 1-separations, combining Props 8.3 + 8.4 we obtain the following characterization of 2-connected matroids:

Prop 8.7: A matroid is 2-connected iff it cannot be expressed as a direct sum of two non-empty matroids.

Recall the earlier examples.

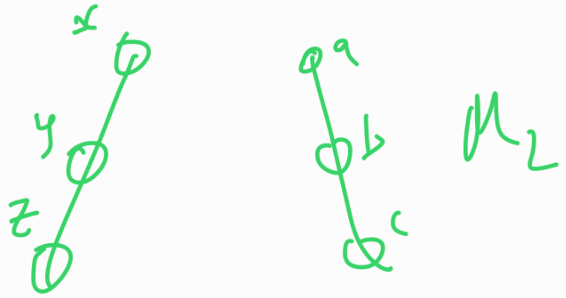


Here

$M_1 =$

$\cong U_{2,3} \oplus U_{2,3}$

On the other hand,



cannot be written as the direct sum of two non-empty matroids.

Note: By submodularity:

$$r(X)+r(E-X) \geq r(M)+r(\phi) = r(M)$$

So $\lambda(X) = r(X)+r(E-X)-r(M) \geq 0$.

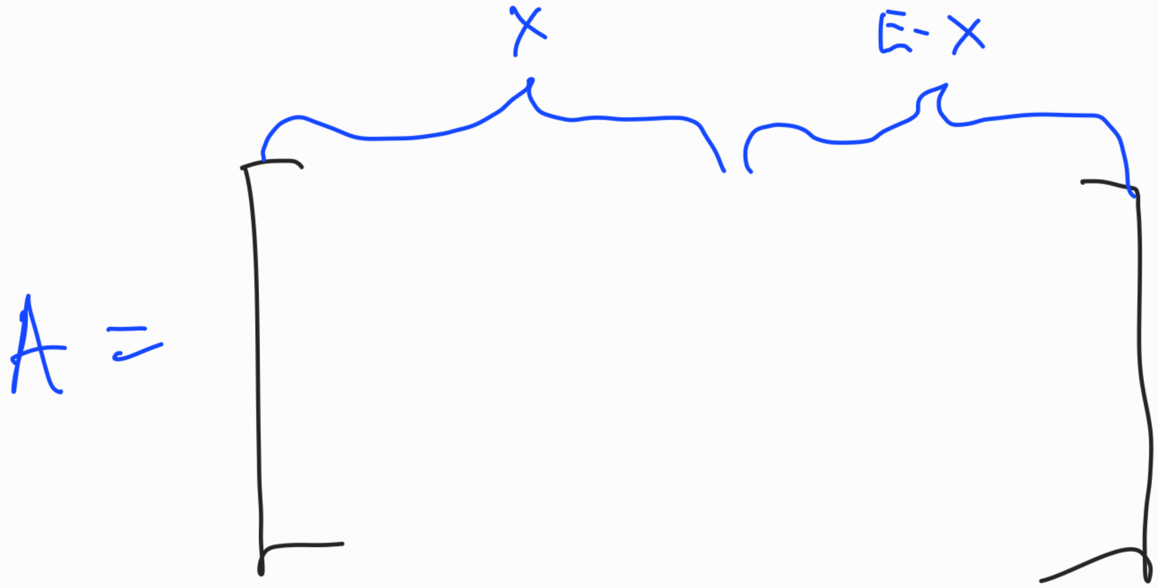
For a 1-separation $(X, E-X)$, $\lambda(X)=0$.

When M is 2-connected (i.e it has no 1-separations) we also just say M is connected.

A geometric interpretation of connectivity

Suppose M is representable over a field $\mathbb{F}$ by a matrix A. Let $(X, E-X)$ be a k-separation of M (for some k).

So $\lambda_M(X) = r(X) + r(E-X) - r(M) < k$.



The vector space spanned by the columns labelled X or $E-X$, are each subspaces of the column space of A.

Call these U and W.

Recall (from linear algebra) that for a vector space V with subspaces U and W

$$\overbrace{\dim(U+W)}^{r(M)} + \overbrace{\dim(U\cap W)}^{\lambda(X)} = \overset{r(\underline{X})}{\dim(U)} + \overset{r(\underline{E}-X)}{\dim(W)}$$

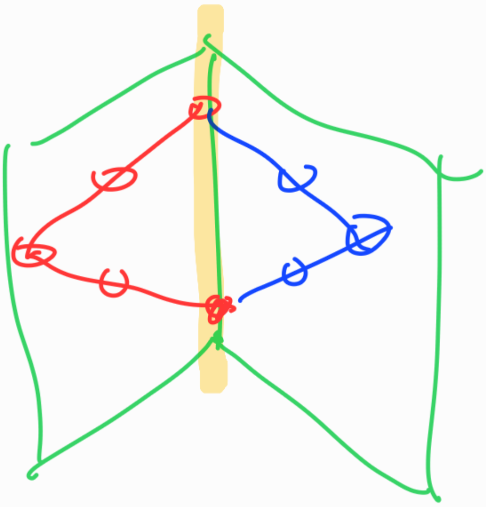
$$\{\underset{\sim}{v} : \underset{\sim}{v} \in U\cap W\}$$

$$U+W = \{\underset{\sim}{u}+\underset{\sim}{w} : \underset{\sim}{u}\in U \text{ and } \underset{\sim}{w}\in W\}$$

It follows that $\lambda(X)$ is the dimension of the intersection of the subspaces U and W.

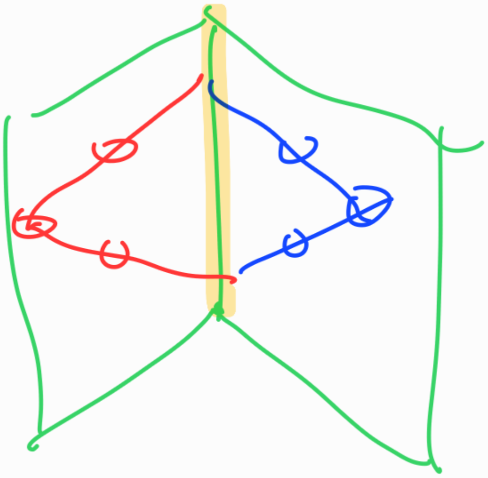
Note however that $r(cl(X)\cap cl(E-X))$ may be smaller than $\lambda(X)$.

Example:



For the partition (X, Y), we have

$$\lambda(X) = 3+3-4 = 2$$



for both examples.

In the first example

$r(cl(X) \wedge cl(E-X)) = 2$

In the second example

$r(cl(X) \wedge cl(E-X)) = 0$