MATH432 | Lecture 15

Assignment 3 due Thursday

Test 27th May

Last time: "cryptomorphisms" i.e.

different, but equivalent ways of defining an object – in this case a matroid.

- bases
- independent sets (satisfying (I1)-(I3))
- circuits
- rank
- closure

→ flats

For $\mathcal{F} \subseteq 2^E$

(F1) $E \in \mathcal{F}$

(F2) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$.

(F3) If $F \in \mathcal{F}$ and $\{F_1, F_2, \dots, F_n\}$ are the minimal members of $\mathcal{F}$ that properly contain F, then $\{F_1 - F, F_2 - F, \dots, F_n - F\}$ is a partition of $E - F$.

Theorem 6.5 If M is a matroid with family of flats $\mathcal{F}$, then $\mathcal{F}$ satisfies (F1)-(F3). Conversely, if E is a set and $\mathcal{F}$ is a family of subsets of E satisfying (F1)-(F3) then there is a matroid on ground set E whose family of flats is $\mathcal{F}$.

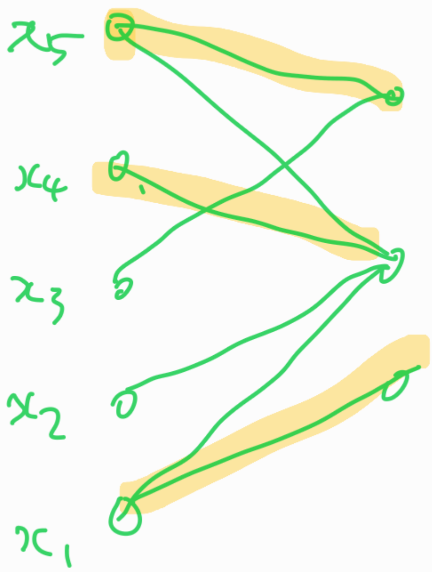
→ independent sets satisfying (I1), (I2), (G1)

For $\mathcal{I} \subseteq 2^E$,

(G1) For all weight functions $w: E \to \mathbb{R}$, the greedy algorithm finds a maximal member of $\mathcal{I}$ of maximum weight.

Theorem 6.6 Let $\mathcal{I}$ be a collection of subsets of a set E. Then $\mathcal{I}$ is the family of independent sets of a matroid on E iff $\mathcal{I}$ satisfies (I1), (I2) and (G1).

Example Consider the transversal matroid on ground set $\{x_1, x_2, \dots, x_5\}$ with the given presentation and weight function $w(x_i) = i$ where we seek a solution of maximum weight.



The greedy algorithm considers...

x_5 after which $X = \{x_5\}$,

then x_4 after which $X = \{x_4, x_5\}$.

Then x_3 — we keep $X = \{x_4, x_5\}$

then x_2 — we keep $X = \{x_4, x_5\}$

then x_1 after which $X = \{x_1, x_4, x_5\}$.

<u>Proof</u> of 6.6 continued from last time.

Last time we saw ($\Rightarrow$). We still need to show that if $\mathcal{I}$ satisfies (I1), (I2), (G1), then we have a matroid whose family of independent sets is $\mathcal{I}$.

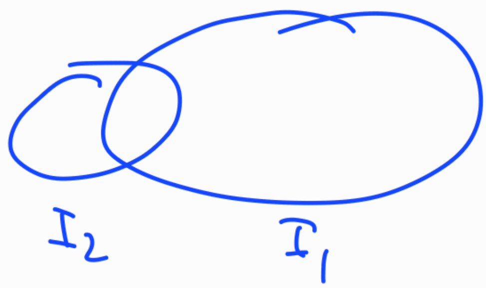
We'll show that if (I3) doesn't hold, then (G1) doesn't hold.

Suppose (I3) fails. Then there exists $I_1, I_2 \in \mathcal{I}$ with $|I_2| < |I_1|$ such that $I_2 \cup e \notin \mathcal{I}$ for all $e \in I_1 - I_2$.

Now $0 \leq |I_2 - I_1| < |I_1 - I_2|$

so we pick $\varepsilon \in \mathbb{R}$ such that

$$\frac{|I_2 - I_1|}{|I_1 - I_2|} < \varepsilon < 1$$



Now we define $w: E \to \mathbb{R}$ as follows:

$$w(e) = \begin{cases} 1 & e \in I_2 \\ \varepsilon & e \in I_1 - I_2 \\ 0 & e \notin I_1 \cup I_2 \end{cases}$$

The greedy algorithm will first pick all elements in I_2. Then it considers the elements in $I_1 - I_2$, but can't add any of these elements to the partial solution X_r, so it will choose a maximal member B_G of $\mathcal{I}$ of weight $|I_2|$.

But, by (I2), I_1 is contained in a maximal member $I_1' \in \mathcal{I}$, and

$$w(I_1') \geq w(I_1) = |I_1 \cap I_2| + \varepsilon |I_1 - I_2|$$

$$> |I_1 \cap I_2| + \frac{|I_2 - I_1|}{\cancel{|I_1 - I_2|}} \cancel{|I_1 - I_2|}$$

$$= |I_1 \cap I_2| + |I_2 - I_1|$$

$$= |I_2| = w(B_G)$$

Thus, the greedy algorithm fails for this weight function, i.e (G1) doesn't hold. □

$\boxed{\text{Representability}}$ Recall:

* a matroid M is $\mathbb{F}$-representable when

$M \cong M[A]$ for some matrix A over $\mathbb{F}$.

* $M[A] = M[A']$ when A' is obtained from A by standard row operations, scaling/swapping columns, applying automorphisms of $\mathbb{F}$.

Constructing a representation Let M be a matroid. Suppose M is $\mathbb{F}$-representable. Then we know M has a $\mathbb{F}$-representation $[\overbrace{I_r}^{B} \mid D]$ in standard form where B labels the columns of I_r, for some basis B of M.

Recall for $y \in E(M) - B$ that $C(y, B)$ is the fundamental circuit of y with respect to B.

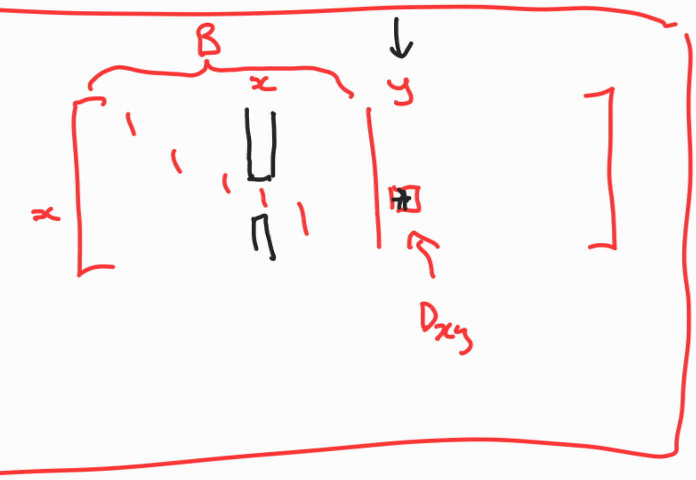
Proposition 7.1: Let $x \in B$ and $y \in E(M) - B$. $D_{xy} \neq 0$ if and only if $x \in C(y, B)$.

Proof: $D_{xy} \neq 0 \iff \det(D[\{x\}, \{y\}]) \neq 0$

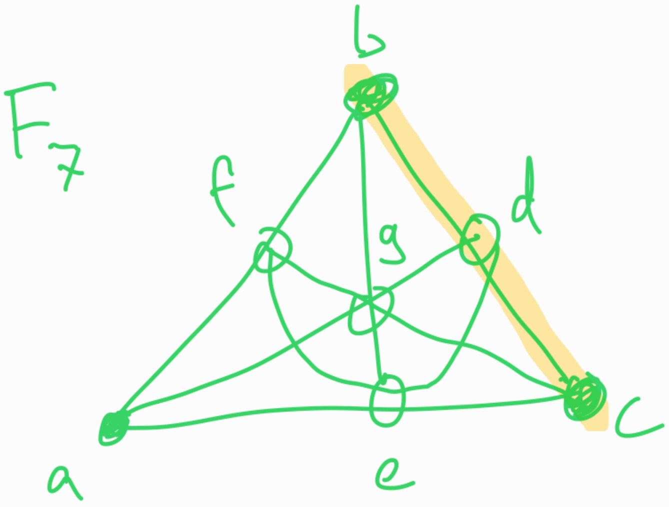
Prop 3.13 $\rightsquigarrow$ $\iff (B-x)\cup y$ is a basis

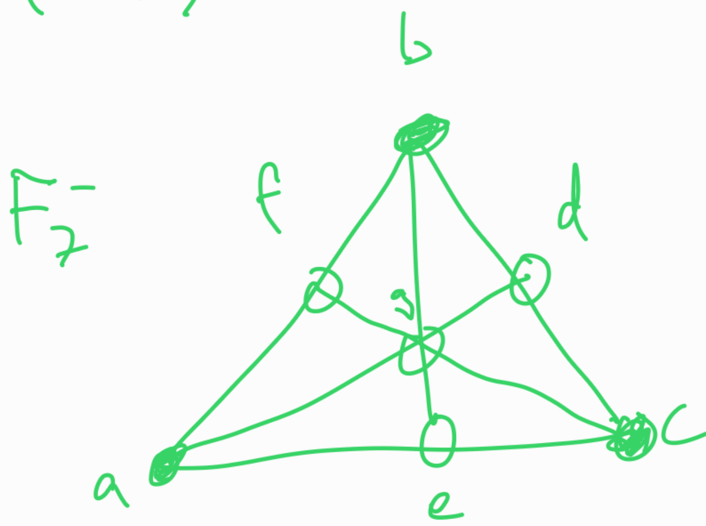
$\iff (B-x)\cup y$ doesn't contain any circuit

$\iff x \in C(y,B)$. $\square$



Example Consider the Fano matroid F_7, and the non-Fano matroid F_7^-.





What fields (if any) are these matroids representable over?

Let $M \in \{F_7, F_7^-\}$

Suppose M is $\mathbb{F}$-representable. Then it has a $\mathbb{F}$-representation of the form:

$$\begin{array}{ccc|cccc} a & b & c & d & e & f & g \end{array}$$
$$\left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & * & D \\ 0 & 0 & 1 & * & \end{array}\right]$$

Consider fundamental circuits with respect to the basis $B = \{a, b, c\}$; these tell us whether entries in D are zero or non-zero.

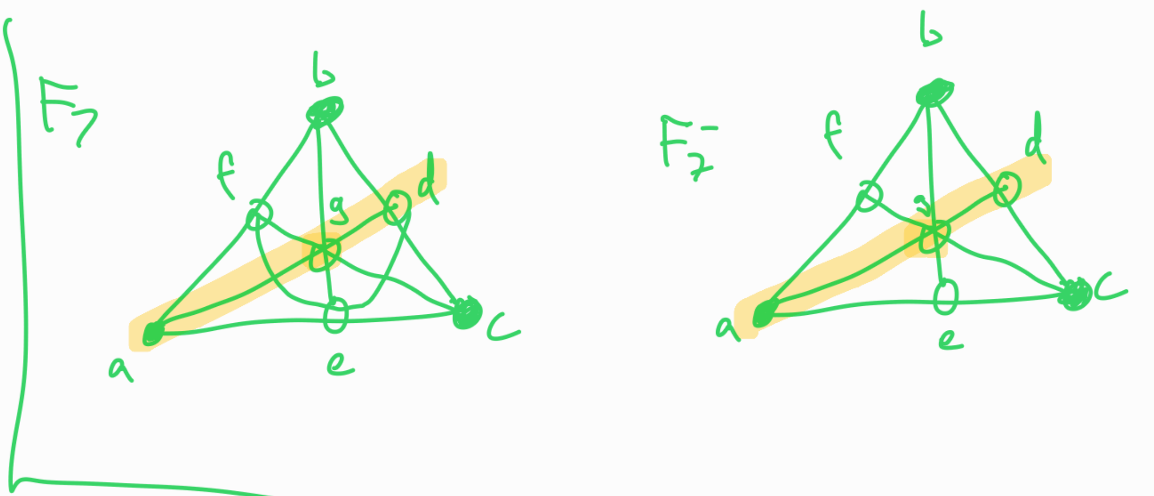
For example, since $\{b, c, d\}$ is a circuit, the column labelled d is of the form $\begin{pmatrix} 0 \\ * \\ * \end{pmatrix}$

where $*$ entries are non-zero. Continuing in this way, we see that if M is $\mathbb{F}$-representable, it has an $\mathbb{F}$-representation of the form

$$\begin{array}{ccc|cccc} a & b & c & d & e & f & g \end{array}$$
$$\left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & * & 0 & * & * \\ 0 & 0 & 1 & * & * & 0 & * \end{array}\right]$$

where $*$s are non-zero.

By rescaling rows and columns, there is an $\mathbb{F}$-representation of the form:



$$
\begin{array}{ccccccc}
a & b & c & d & e & f & g \\
\end{array}
$$

$$
\left[\begin{array}{ccc|cccc}
1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & z & 1 \\
0 & 0 & 1 & x & y & 0 & 1
\end{array}\right]
$$

where x, y and z are non-zero.

Since $\{a,d,g\}$ is dependent, $\begin{vmatrix} 1 & 1 \\ x & 1 \end{vmatrix} = 0$, so $x=1$.

Since $\{b,e,g\}$ is dependent, $\begin{vmatrix} 1 & 1 \\ y & 1 \end{vmatrix} = 0$, so $y=1$

Since $\{c,f,g\}$ is dependent, $\begin{vmatrix} 1 & 1 \\ z & 1 \end{vmatrix} = 0$, so $z=1$.

When $\underline{M = F_7}$, $\{d,e,f\}$ is dependent, so

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 1 + 1$$

$$= 0,$$

so this matroid can only be represented over fields with characteristic 2 ($GF(2)$ or $GF(2^k)$ for $k > 1$)

After checking the other 3-element sets, we see that we have constructed a $\mathbb{F}$-representation for F_7 provided $\mathbb{F}$ has characteristic 2.

For the non-Fano matroid,

$$A = \left[I \;\middle|\; \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{matrix} \right]$$ is an $\mathbb{F}$-representation

provided $\mathbb{F}$ has characteristic not 2.