Assignment 3 due Thursday 2 May

Last time: closure

$$cl(X) = \{e \in E(M) : r(X \cup e) = r(X)\}$$

flats

a set X such that $cl(X) = X$.

Prop 5.9 For $X \subseteq E(M)$ and $e \in E(M) - X$

$e \in cl(X)$ iff there exists a circuit C such that $e \in C$ and $C \subseteq X \cup e$.

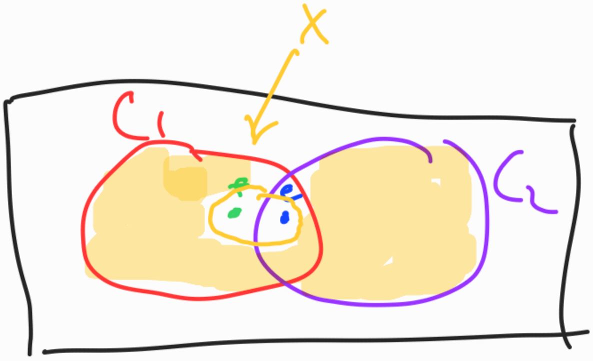
Strong circuit elimination:

(C3') If $C_1, C_2 \in \mathcal{C}$, $e \in C_1 \cap C_2$ and $f \in C_1 - C_2$ then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$ and $f \in C_3$.

Prop 5.13 Let M be a matroid. Then $\mathcal{C}(M)$ satisfy (C3').

Proof: Let $C_1, C_2 \in \mathcal{C}(M)$ and $e \in C_1 \cap C_2$ and $f \in C_1 - C_2$.

Let $X = (C_1 \cup C_2) - \{e, f\}$.



As $C_2 \subseteq X \cup e$ and $e \in C_2 - X$, we have, by Prop 5.9, that $e \in cl(X)$.

Similarly, $C_1 \subseteq (X \cup e) \cup f$ and $f \in C_1 - (X \cup e)$

so, by Prop 5.9 again, $f \in cl(X \cup e)$.

Now $r(X \cup \{e, f\}) = r(X \cup e) = r(X)$ and

$r(X) \leq r(X \cup f) \leq r(X \cup \{e, f\}) = r(X)$

So $r(X \cup f) = r(X)$, i.e. $f \in cl(X)$.

By Prop 5.9 there exists a circuit containing f and contained in $X \cup f = (C_1 \cup C_2) - e$,

as required. □

Cryptomorphisms

We've discussed several different, equivalent, ways of characterising a matroid, including:

bases, independent sets, circuits, the rank function, the closure operator, flats.

The various equivalent axiomatic schemes are sometimes said to be cryptomorphic.

Theorem 6.1 Let E be a set.

If $\mathcal{B}$ is a family of subsets of E satisfying (B1) and (B2), and

$\mathcal{I} = \{I \subseteq E : I \subseteq B \text{ for some } B \in \mathcal{B}\}$ then $\mathcal{I}$ satisfies (I1) - (I3).

Conversely, if $\mathcal{I}$ satisfies (I1)-(I3) and $\mathcal{B}$ consists of the maximal members of $\mathcal{I}$, then $\mathcal{B}$ satisfies (B1) and (B2).

Theorem 6.2: If M is a matroid, then $\mathcal{C}(M)$ satisfies (C1)-(C3). Conversely if $\mathcal{C}$ satisfies (C1)-(C3), then there is a matroid on E whose circuits are $\mathcal{C}$ and whose independent sets are:

$$\{X \subseteq E : \text{for each } C \in \mathcal{C}, \text{ we have } C \not\subseteq X\}.$$

Theorem 6.3 If M is a matroid with rank function r, then r satisfies (R1)-(R3). Conversely if $r: 2^E \to 2^E$ is a function that satisfies (R1)-(R3), then there is a matroid on E whose rank function is r and whose independent sets are:

$$\{X \subseteq E : r(X) = |X|\}.$$

Theorem 6.4 If M is a matroid with closure operator cl, then cl satisfies (CL1)-(CL4). Conversely if E is a set and $cl: 2^E \to 2^E$ satisfies (CL1)-(CL4), then cl is the closure operator of a matroid on E, whose independent sets are:

$$\{I \subseteq E : e \notin cl(I - e) \text{ for each } e \in I\}.$$

Now for a new cryptomorphism, using flats.

Let $\mathcal{F}$ be a family of subsets of a set E, and consider the properties:

(F1) $E \in \mathcal{F}$.

(F2) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$.

(F3) If $F \in \mathcal{F}$ and $\{F_1, F_2, \ldots, F_n\}$ are the minimal members of $\mathcal{F}$ that properly contain F, then $(F_1 - F, F_2 - F, \ldots, F_n - F)$ is a partition of $E - F$.

Theorem 6.5. If M is a matroid with family of flats $\mathcal{F}$, then $\mathcal{F}$ satisfies (F1)–(F3). Conversely, if E is a set and $\mathcal{F}$ is a family of subsets satisfying (F1)–(F3), then there is a matroid on ground set E whose family of flats is $\mathcal{F}$.

Kruskal's algorithm is a method for finding a maximum-weight spanning tree in a connected edge-weighted graph.

A weight function $w: E(G) \to \mathbb{R}$ assigns each edge a weight. For $X \subseteq E(G)$, the weight of X is $\sum_{x \in X} w(x)$.

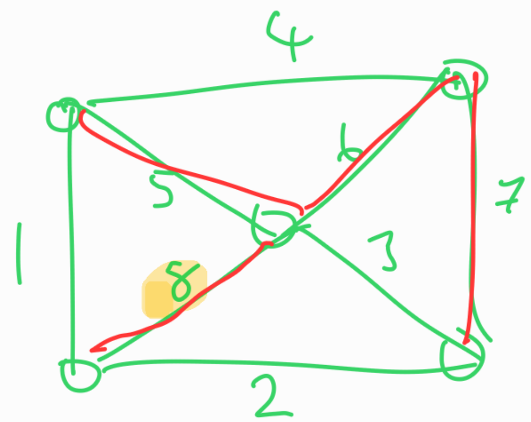
The algorithm is:

Initially set $X := \emptyset$

While possible:

- choose an edge e of maximum weight such that $X \cup e$ doesn't contain a cycle
- set $X := X \cup e$

e.g.



$X = \emptyset$

$X = \{8\}$

$X = \{8, 7\}$

$X = \{8, 7, 6\}$

$\rightarrow X = \{8, 7, 6, 5\}$

This is said to be a "greedy algorithm" since at each step, we pick what appears to be the best option.

Input: a matroid on ground set E, and a weight function $w: E \to \mathbb{R}$.

Problem: find a maximal member of $\mathcal{I}(M)$ of maximum weight.

We define the greedy algorithm to solve this problem, as follows:

Initially set $X := \emptyset$

While possible:

- choose $e \in E(M)$ of maximum weight such that $X \cup e \in \mathcal{I}(M)$

- set $X := X \cup e$

Note that Kruskal's algorithm on a graph G can be viewed as the greedy algorithm on $M(G)$.

Consider the following property for a family of subsets $\mathcal{I}$ of E:

(G1) For all weight functions $w: E \to \mathbb{R}$, the greedy algorithm finds a maximal member of $\mathcal{I}$ of maximum weight.

<u>Theorem 6.6</u>: Let $\mathcal{I}$ be a collection of subsets of a set E. Then $\mathcal{I}$ is the family of independent sets of a matroid on E iff $\mathcal{I}$ satisfies (I1), (I2), (G1).

<u>Proof</u>: ($\Rightarrow$) Let M be a matroid and $\mathcal{I} = \mathcal{I}(M)$. We've seen that (I1) and (I2) hold, so it suffices to show (G1) holds.

Suppose B_G is a basis of M corresponding to a solution found by the greedy algorithm, and let B be another basis of M. Let

$B_G = \{e_1, e_2, \ldots, e_r\}$ such that for all i

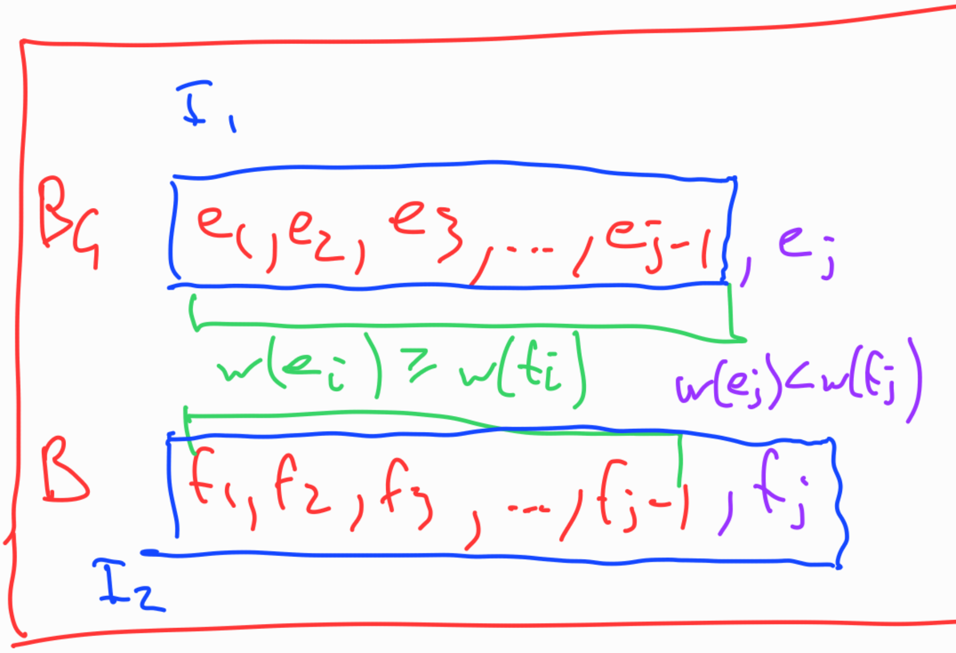
$B = \{f_1, f_2, \ldots, f_r\}$ $w(e_i) \geq w(e_{i+1})$ and $w(f_i) \geq w(f_{i+1})$

i.e non-increasing weight order.

We claim $w(e_i) \geq w(f_i)$ for each i.

Suppose not, and let j be the smallest index for which $w(e_j) < w(f_j)$.

Then $I_1 = \{e_1, e_2, \dots, e_{j-1}\}$ and $I_2 = \{f_1, \dots, f_{j-1}, f_j\}$ are in $\mathcal{I}$, and $|I_1| < |I_2|$, so by (I3) there exists $f \in I_2 - I_1$ such that $I_1 \cup f \in \mathcal{I}$.



But $w(f) \geq w(f_j) > w(e_j)$, so the greedy algorithm would choose f rather than e_j, contradicting that $e_j \in B_G$.

Now $w(e_i) \geq w(f_i)$ for all i, implying $w(B_G) \geq w(B)$. Since B was chosen arbitrarily, (G1) holds.