MATH4321 Lecture 11

Assignment 2 due 5pm

Recap: When e is not a loop:

$$\mathcal{I}(M/e) = \{ I \subseteq E(M)-e : I \cup e \in \mathcal{I}(M) \}$$

Otherwise: $\mathcal{I}(M/e) = \mathcal{I}(M\backslash e)$.

(*) $$\mathcal{B}(M/e) = \{ B \subseteq E(M)-e : B \cup e \in \mathcal{B}(M) \}$$

$\mathcal{C}(M/e)$ consists of the minimal members of

$$\{ C-e : C \in \mathcal{C}(M) \}$$

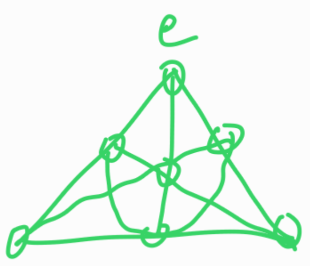
$$r_{M/e}(X) = r_M(X \cup e) - r_M(\{e\})$$

Geometric representations of minors:

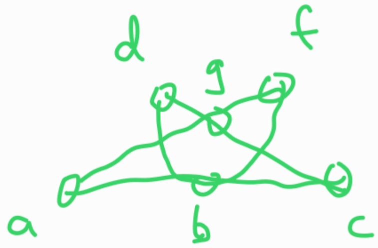
$$\mathcal{I}(M\backslash e) = \{ I \subseteq \mathcal{I}(M) : e \notin I \}$$

* to delete an element e, simply remove the point e from the representation.

e.g.

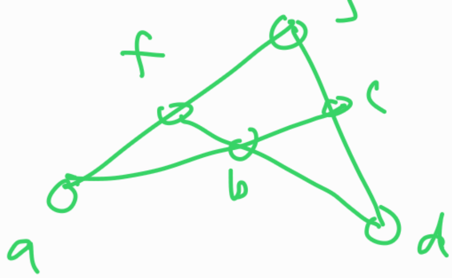


F_7

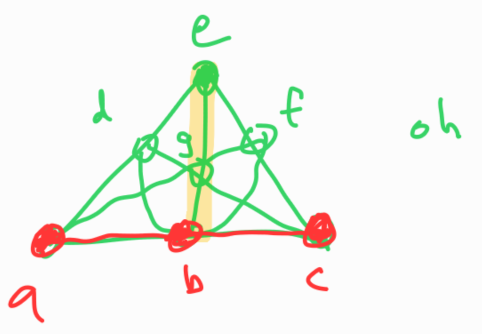


$F_7 \backslash e$

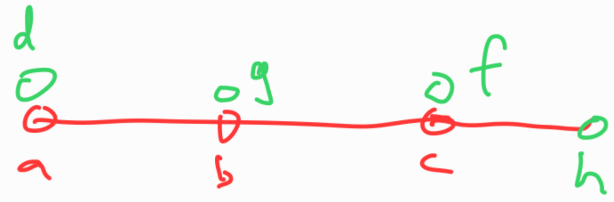
which is the same as



Now, contraction.



M



M/e

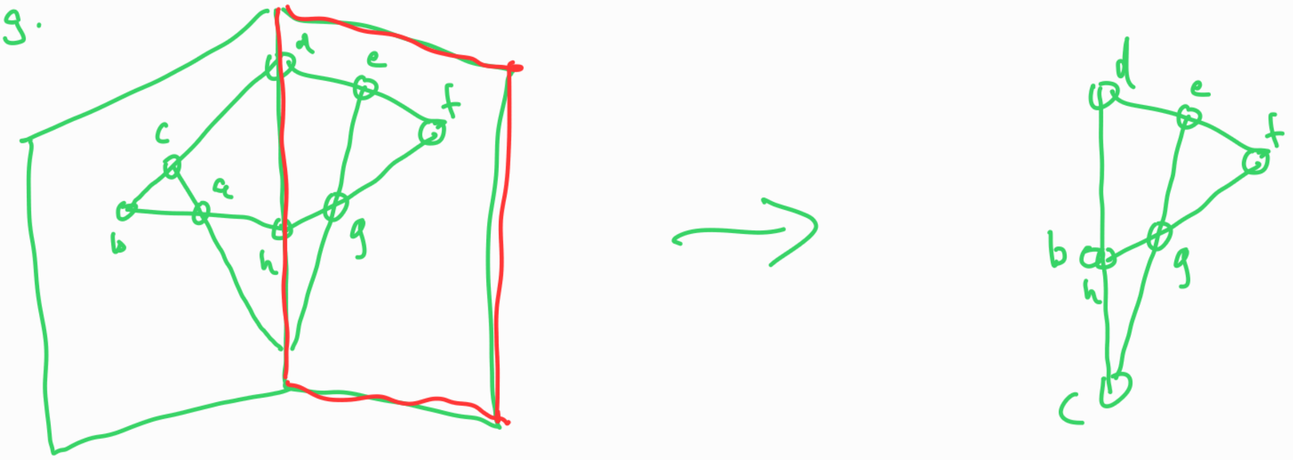
Consider a hyperplane H that avoids e

For a maximal independent set I contained in H

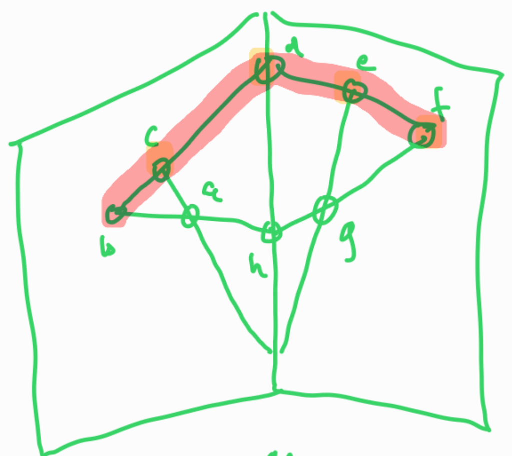
$I \cup e$ is a basis of M

so I is a basis of M/e

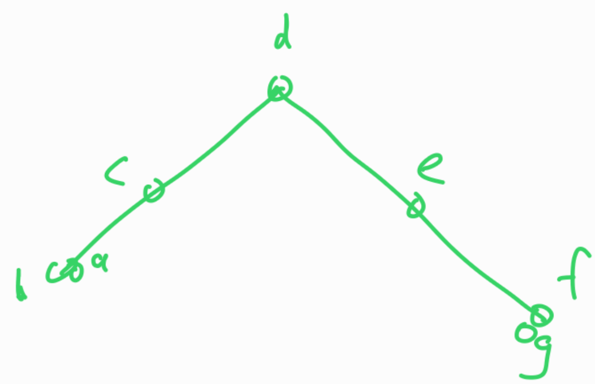
e.g.



M M/a



M



M/h

Minors and duality:

Proposition: Let G be a connected plane graph.
Let e be a non-loop edge of G.
Then $(G/e)^*$ is isomorphic to $G^* \backslash e$

Proposition 4.18: Let M be a matroid and $e \in E(M)$

Then (i) $(M/e)^* = M^* \backslash e$, and

(ii) $(M\backslash e)^* = M^*/e$.

Proof: Let $E = E(M)$.
We first show (i) holds when e is not a loop.
So let e be a non-loop element of M.
Let $B^* \subseteq E - e$. Then

B^* is a basis of $(M/e)^* \iff (E-e)-B^*$ is a basis of M/e

(defn of duality)

$\iff E-B^*$ is a basis of M containing e

$\iff B^*$ is a basis of M^* not containing e

Prop 4.13 (i)

$\iff B^*$ is a basis of $M^* \backslash e$

This shows (i) holds when e is not a loop.

Next we show (ii) holds when e is not a coloop.

Suppose e is not a coloop of M. Then e is not a loop of M^*.

Applying (i) to M^*,

$$(M^*/e)^* = (M^*)^* \backslash e = M \backslash e, \text{ so}$$

$$M^*/e = ((M^*/e)^*)^* = (M \backslash e)^*$$

so (ii) holds when e is not a coloop.

Now assume e is a loop in M. Then $M \backslash e = M/e$.

$$\text{So} \quad (M \backslash e)^* = (M/e)^*$$

Note that e is not also a coloop in M (since e is a loop if is not in any basis, so in particular there is a basis that doesn't contain e). So

$$M^*/e = (M \backslash e)^*$$

(Prop 4.4)

Since e is a coloop of M^*, $\quad M^* \backslash e = M^*/e$

$$M^* \backslash e = M^*/e = (M \backslash e)^* = (M/e)^*$$

So $M^* \backslash e = (M/e)^*$ when e is a loop.

We still need to show (ii) holds when e is a coloop.

By applying (i) to the matroid M^* we see that

$$(M^*/e)^* = (M^*)^* \backslash e = M \backslash e$$

so by duality $M^*/e = (M \backslash e)^*$ as req^d. $\square$

<u>Corollary</u> 4.19: Let e be an element in a matroid M.

Then (i) $M/e = (M^* \backslash e)^*$, and

(ii) $M \backslash e = (M^*/e)^*$.

<u>Proposition</u> 4:21: For distinct elements e and f in a matroid M

i) $(M \backslash e) \backslash f = (M \backslash f) \backslash e$

ii) $(M/e)/f = (M/f)/e$

iii) $(M \backslash e)/f = (M/f) \backslash e$

Proof in online notes.

Hence, for a matroid M and set $X \subseteq E(M)$, we can unambiguously write $M \backslash X$ to denote the deletion of each element in X (and similarly for M/X).

<u>Corollary</u> 4:23: Let N be a minor of a matroid M. Then $N = M \backslash X / Y$ for some coindependent set X and some independent set Y.

__Def__: For a matroid M with ground set E, and $X \subseteq E$, the __restriction__ of M to X, denoted $M|X$, is $M \setminus (E-X)$

Note that for a matroid M and set $X \subseteq E(M)$, the bases of $M|X$ are the maximal independent sets of M contained in X. For this reason, a basis of $M|X$ is also sometimes called a __basis of X__.