MATH432 Lecture 1

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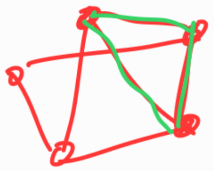
office hour 4:10 Tues

We use

graphs } to describe { binary relationships



groups — symmetries

topological spaces — continuous deformations

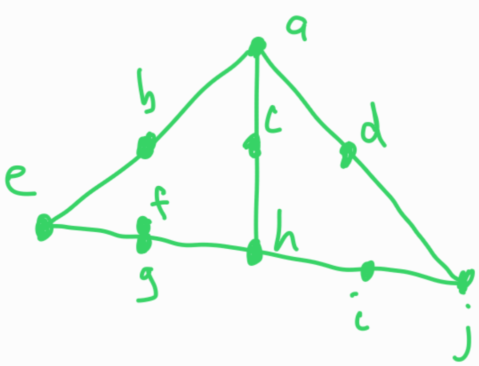
metric spaces — objects with a notion of distance

Similarly, matroids capture a notion of independence or dependence

This notion underlies

* linear independence for a set of vectors in a vector space
* the "independence" of a set of edges in a graph that don't contain a cycle.
* geometric independence

Example 1: Consider the following points in $\mathbb{R}^2$



Note: (i) f and g are copunctual (lie at the same point)

(ii) $\{a,b,e\}$ are collinear, i.e. they lie on a line.

$\{b,c,d\}$ are not collinear

$\{c,f,g\}$ are collinear

We will define some subsets of points to be independent.

For this example in $\mathbb{R}^2$, we first consider sets of size 3 – we say a set of 3 points is a basis if these points are not collinear.

e.g. $\{b,c,d\}$ is a basis

$\{a,d,j\}$ is not a basis.

We say a set of points is independent if it is contained in a basis, otherwise it is dependent.

$\{f,g\}$ is dependent, all other sets of size 2 are independent.

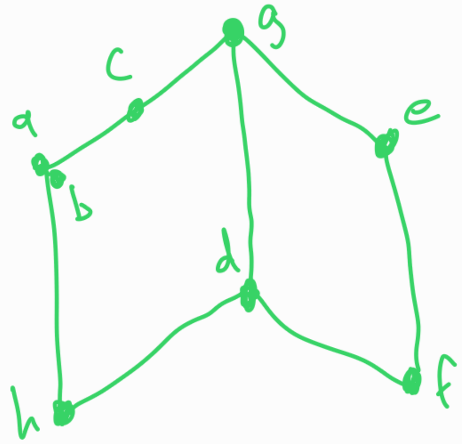
$\{b,c,d\}$ is independent

any set of size 4 is dependent.

When considering points in $\mathbb{R}^2$, a basis has size 3

$\mathbb{R}^n$, a basis has size $n+1$

Example 2: Consider the following points in $\mathbb{R}^3$



$\{a,c,h,f\}$ is a basis since they are not contained in a plane

A $\left\{\begin{array}{l}\text{pair of points}\\ \text{3-elt subset of points}\\ \text{4-elt subset of points}\end{array}\right\}$ is dependent $\left\{\begin{array}{l}\text{if they are copunctual}\\ \text{if they are collinear}\\ \text{if they are coplanar}\end{array}\right.$

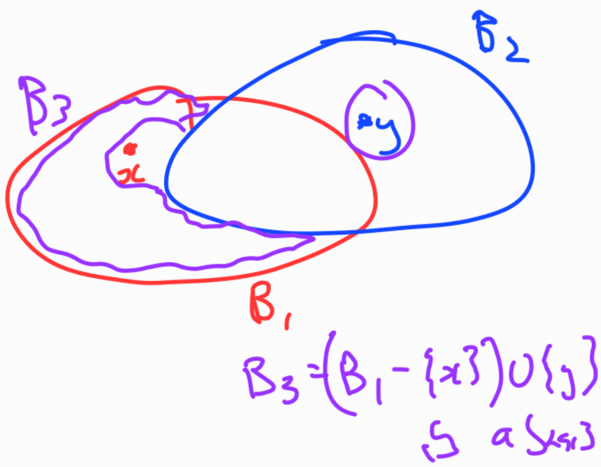
Any set of 5 or more points is also dependent.

Any other set is independent.

In both examples, and in fact for a configuration of points in $\mathbb{R}^n$, the following property holds:

If B_1 and B_2 are bases, and $x \in B_1 - B_2$, then there exists an element $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\}$ is a basis.

We call this the basis exchange property



<u>Definition</u> (matroid): A <u>matroid</u> is a pair $(E, \mathcal{B})$ where E is a finite set, and $\mathcal{B}$ is a family of subsets of E satisfying

(B1) $\mathcal{B} \neq \emptyset$

(B2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, then there exists an element $B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$.

We call E the ground set of the matroid, and each $B \in \mathcal{B}$ is called a basis.

Let $M = (E, \mathcal{B})$ be a matroid. A set $X \subseteq E$ is independent if $X \subseteq B$ for some $B \in \mathcal{B}$; otherwise, X is dependent.

Proposition 1.3: Let M be a matroid with ground set E and bases B_1 and B_2. Then $|B_1| = |B_2|$

Proof: Suppose not. Choose bases B_1 and B_2 such that $|B_1| > |B_2|$, and maximising $|B_1 \cap B_2|$. (*)

Since $|B_1| > |B_2|$ there exists $x \in B_1 - B_2$, so by (B2), there exists $y \in B_2 - B_1$ such that

$$B_3 = (B_1 - \{x\}) \cup \{y\} \text{ is a basis of } M.$$

Then
$$\begin{aligned} B_3 \cap B_2 &= ((B_1 - \{x\}) \cup \{y\}) \cap B_2 \\ &= ((B_1 - \{x\}) \cap B_2) \cup \{y\} \\ &= (B_1 \cap B_2) \cup \{y\} \end{aligned}$$

so $|B_3 \cap B_2| > |B_1 \cap B_2|$, contradicting $(*)$. $\square$

Note: for sets X and Y, we use $X-Y$ to denote set difference i.e. $X-Y = \{e \in X : e \notin Y\}$.

We let $\mathcal{I}(M)$ denote the independent sets of a matroid M.

What properties does $\mathcal{I}(M)$ satisfy?

Consider the following properties for a family of subsets $\mathcal{I}$ of a set E.

(I1) $\emptyset \in \mathcal{I}$

(I2) If $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$, then $I_2 \in \mathcal{I}$.

(I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_2| < |I_1|$, then there exists $e \in I_1 - I_2$ such that $I_2 \cup \{e\} \in \mathcal{I}$.

(I3) is called the independence augmentation property.

Thm 1.7: If M is a matroid whose family of independent sets is $\mathcal{I}$, then $\mathcal{I}$ satisfies (I1)-(I3).

Conversely, if E is a set and $\mathcal{I}$ is a family of subsets of E satisfying (I1)-(I3), then there is a matroid

$M = (E, \mathcal{B})$ whose independent sets are $\mathcal{I}$ where $\mathcal{B}$ consists of the maximal members of $\mathcal{I}$.

Uniform matroids

Let E be a set of size n.

For an integer r such that $0 \leq r \leq n$, let

$$\mathcal{B} = \{ B \subseteq E : |B| = r \}.$$

Then $U_{r,n} = (E, \mathcal{B})$ is a matroid.

Exercise 1: Prove $U_{r,n}$ is a matroid.

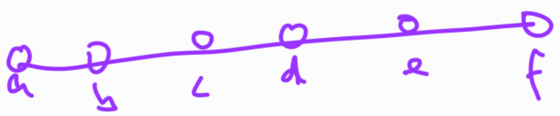
e.g. 1. Consider $U_{1,2}$ on ground set $\{a, b\}$.

Then the bases are $\mathcal{B} = \{\{a\}, \{b\}\}$.

independent sets are $\mathcal{I} = \{ \{a\}, \{b\}, \emptyset \}$.

is a geometric representation of $U_{1,2}$.

e.g. 2 Consider $U_{2,6}$ on ground set $\{a, b, c, d, e, f\}$.



is a geometric repⁿ of $U_{2,6}$.

Exercise 2: draw a geometric repⁿ for $U_{3,5}$.