Name:________________________ ID number:________________________

- Duration: 50 MINUTES. 50 Marks
- There are FIVE questions, on FIVE pages. Attempt every question in the spaces provided. Use the reverse side if you run out of space.
- Write your name and ID number in the spaces provided.

Question 1. (9 marks)

(a) State the Handshaking Lemma. [2]

(b) Let G be a connected graph. Give the definition of a *spanning tree* of G. [1]

(c) Let G be a graph. Prove that G is a forest if and only if every edge of G is a bridge. [6]

Question 2. (12 marks)

(a) Let G be a graph. Give the definition of a *vertex cut* of G. [1]

(b) Let G be a simple graph. Give the definition of a *block* of G. [2]

(c) Let G be a graph. Are the following statements true or false? Justify your answer with an explanation if true, or give a counterexample if false.

(i) If H is a subgraph of G, then H is an induced subgraph of G. [3]

(ii) If H is an induced subgraph of G, then H is a minor of G. [3]

(iii) If G is simple and 2-connected, then G has exactly one block. [3]

Question 3. (8 marks)

(a) By drawing an appropriate graph, give a clearly illustrated example of the following:

- a graph with exactly one cut vertex and exactly one bridge. [2]

(b) Let G be a 3-connected graph with a vertex v. Prove that $G - v$ is 2-connected. [6]

Question 4. (14 marks)

(a) Are the following statements true or false? Justify your answer with an explanation if true, or give a counterexample if false.

(i) If G is a 3-connected graph with at least five vertices, and e is an edge of G, then G/e is 3-connected. [3]

(ii) For every integer $k \geq 2$, there exists a k-connected graph with precisely $k+1$ vertices. [3]

(b) Let G be a graph and let X and Y be subsets of $V(G)$. Menger's Theorem states that the maximum number of vertex-disjoint (X,Y)-paths is equal to the minimum order of a separation that separates X from Y.

(i) Define what is meant by an (X,Y)-*path*. [1]

(ii) Define what it means for a separation $\{A,B\}$ to *separate* X *from* Y. [1]

(iii) Explain why it follows from Menger's theorem that if G is 2-connected and X and Y each have size two, then there are two vertex-disjoint (X,Y)-paths in G. [6]

Question 5. (7 marks)

(a) Define a *plane graph* (you may make reference to a *planar embedding* without defining this term). [1]

(b) Define the *degree* of a face in a plane graph. [1]

(c) By drawing an appropriate graph, give clearly illustrated examples of the following:

(i) A connected plane graph with a face whose boundary is not a cycle. [2]

(ii) A plane graph G such that $(G^*)^* \neq G$. [3]

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