Name:____________________ ID number:____________________

- Duration: 50 MINUTES. 40 Marks
- Write your name and ID number in the spaces provided.
- Attempt every question.

**Question 1.** (i) Define what is meant by a *bipartite* graph. [1]

(ii) State the Handshaking Lemma. [1]

(iii) Recall the complete bipartite graph $K_{mn}$. Use the Handshaking Lemma to prove that $K_{mn}$ has $mn$ edges. [2]

(iv) Prove that if $C$ is a cycle of a bipartite graph, then $C$ has an even number of edges. [3]

**Question 2.** Let $G$ be a connected graph.

(i) Define what is meant by an *isthmus* of $G$. [1]

(ii) Let $e = uv$ be an isthmus of $G$. Prove that $u$ and $v$ lie in different components of $G\backslash e$. [3]

(iii) Define what is meant by a *cut vertex* of $G$. [1]

(iv) Draw a clearly labelled graph that has *exactly one* isthmus and *exactly one* cut vertex. [3]

**Question 3.** Let $e = uv$ be an edge of the graph $G$ and let $C$ be a cycle of the graph $G$.

(i) Prove that, if $e \in C$, then $C - \{e\}$ is a cycle of $G/e$. [3]

(ii) Assume that $e \notin C$, but that both $u$ and $v$ are vertices of $C$. Prove that $C$ is *not* a cycle of $G/e$. [2]

**Question 4.** (i) Define what it means for a graph $G$ to be 2-connected. [2]

(ii) Draw a clearly labelled 2-connected graph with two clearly labelled edges $e$ and $f$ having the property that

- $G\backslash e$ is not 2-connected; and [2]
- $G/f$ is not 2-connected. [2]

**Question 5.** Recall the following theorem from the notes. Let $G$ be a *loopless* graph, with *at least two edges* and *no isolated vertices*. Then $G$ is 2-connected if and only if, for any pair $a, b$ of edges, $G$ has a cycle containing both $a$ and $b$.

(i) Explain why we need the condition that $G$ is *loopless*. [1]

(ii) Explain why we need the condition that $G$ has *at least two edges*. [1]

(iii) Explain why we need the condition that $G$ has *no isolated vertices*. [1]

(iv) Use the theorem to prove that if $u$ and $v$ are distinct *vertices* of a 2-connected graph $G$, then $G$ has a cycle containing both $u$ and $v$. [3]

**Question 6.** Let $G = (V, E)$ be a graph.

(i) Define what is meant by a *separation* in $G$. [2]

(ii) Define what is meant by the *boundary* of a separation in $G$. [1]

(iii) Define what is meant by the *order* of a separation in $G$. [1]

**Question 7.** Consider the graph $G$ illustrated below.

(i) Find all proper separations of order 1 in $G$. [3]

(ii) Find a proper separation of order 2 in $G$. [1]