MATH361 | Lecture 7

* assignment 1 due 3pm tomorrow

* course notes on connectivity online

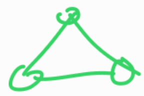
Last time:

cut vertex: vertex v such that $G-v$ has more components than G

2-connected graph: connected graph with no cut vertices and at least 3 vertices

e.g. K_2  is not 2-connected

K_3 is 2-connected



k-vertex cut: a set of vertices X with $|X|=k$ such that $G-X$ has more components than G

k-connected graph: connected graph with no j-vertex cuts for $j<k$ and at least $k+1$ vertices

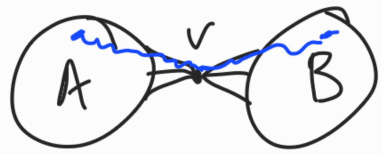
e.g. K_k is not k-connected

K_{k+1} is k-connected

A cut vertex v in a connected graph shows that the graph is connected only in a fragile way

1) we can delete a single vertex v to disconnect it

2) there is a bottleneck: certain paths (from A to B in the illustration)



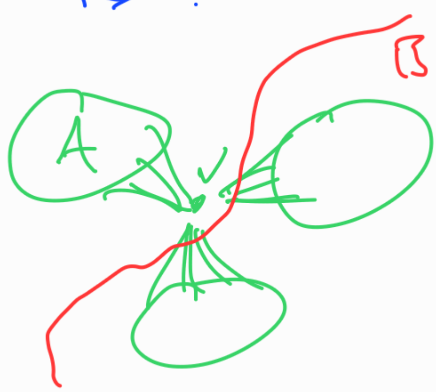
must pass through v.

later we'll see Menger's theorem which generalises this idea.

Lemma 2.19: Let G be a connected graph with a vertex v. Then v is a cut vertex if and only if there is a partition $\{A, B\}$ of $V(G) \setminus \{v\}$ such that every path from a vertex in A to a vertex in B passes through v.

Note: a partition $\{X_1, \dots, X_n\}$ (typically) requires that each X_i is non-empty. Here $A \neq \emptyset$ and $B \neq \emptyset$.

Proof of ($\Rightarrow$): Suppose v is a cut vertex. Then $G - v$ is disconnected.



Let A be the vertex set of one component of $G-v$, and let $B = V(G-v) \setminus A$, so $\{A, B\}$ is a partition of

$V(G) \setminus \{v\}$. Then, for every $a \in A$ and $b \in B$ that there is no path from a to b in $G - v$. But G is connected, so there is a path from a to b in G. Therefore, every path from a to b passes through v.

($\Leftarrow$) left as an exercise. $\square$

It is useful to be able to delete/contract an edge and maintain connectivity properties.

Suppose G is connected and $e \in E(G)$

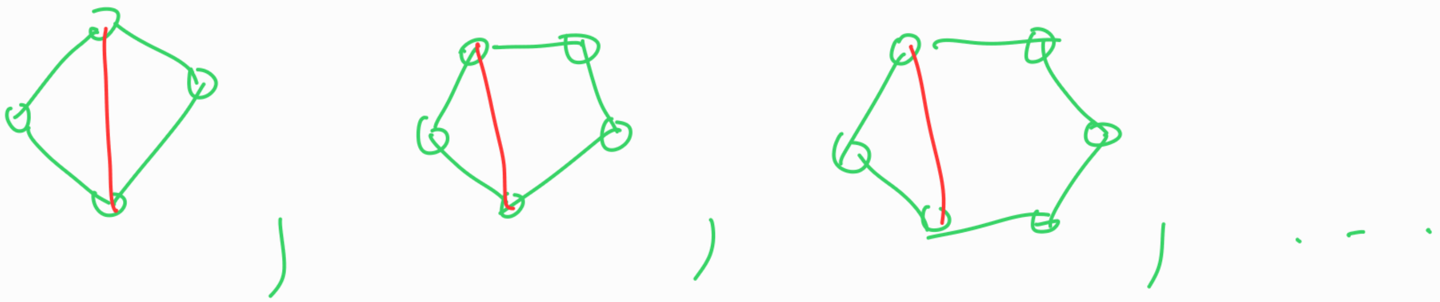
$$G \setminus e \text{ is connected} \iff e \text{ is not a bridge}$$

(When G is connected, there exists an edge e such that $G \setminus e$ is connected $\iff$ G is not a tree)

<u>Lemma</u> 3.1: Suppose G is connected and $e \in E(G)$. Then G/e is connected.

Proof left as an exercise.

What about if G is 2-connected - can we always delete/contract some edge and stay 2-connected?

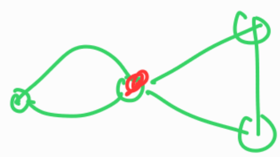


G_1 G_2 G_3

These graphs are all 2-connected

However after deleting any green edge the graph is no longer 2-connected, and after contracting a red edge, the graph is not 2-connected.

e.g. G_3/e



However...

<u>Theorem</u> 3.4: Let G be a 2-connected graph with $|V(G)| \geq 4$, and $e \in E(G)$. Then at least one of G/e and $G \setminus e$ is 2-connected.

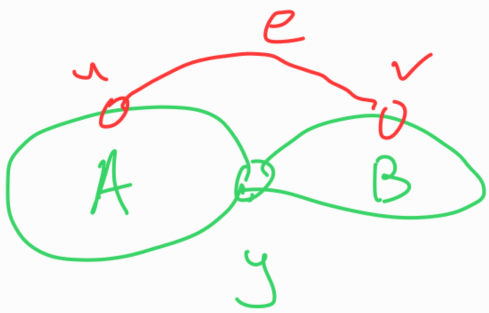
<u>Proof</u> (sketch)

Our strategy to prove this is as follows.

Step 1:

Suppose $G\backslash e$ is not 2-connected.

Then we have:



$G\backslash e$ has a cut vertex y and we can use Lemma 2.19.

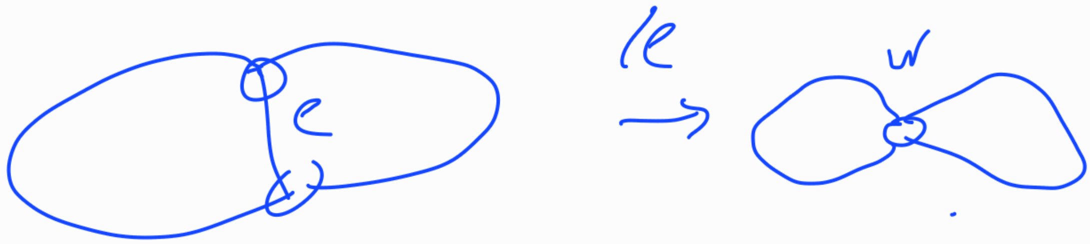
Consider G/e, where w is the vertex resulting from the contraction of $e = uv$.

Step 2a: Show that w is not a cut vertex of G/e

i.e. show that $(G/e) - w$ is connected

(full details in 3.4.2).

Step 2b: Show that if G/e is not 2-connected then w is a cut vertex in G/e.



(full details in 3.4.1)