MATH361 | Lecture 30

Assignment 5 due 28 May

exam 10 June

Last days for course/teaching surveys

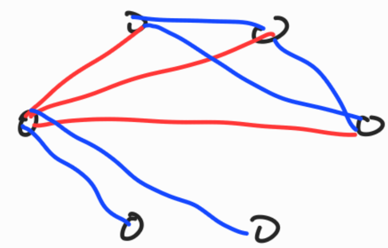
Recap: Ramsey Theory

Theorem 7.1 (Ramsey's Theorem)

For every positive integer t, there exists a number $r(t)$ such that if G is a 2-edge-coloured complete graph with $|V(G)| \geq r(t)$, then G has a monochromatic clique of size t.

e.g. when $t=3$, there exists a number n such that any any 2-edge-colouring of K_n has a monochromatic triangle.

From an assignment question, we saw $n=6$ works.



Lexicographic orderings provide a way to impart a linear order on words/vectors.

We'll just need an ordering on pairs (s,t).

Theorem 7.2

For all positive integers s, t, there exists a number $r(s,t)$ such that if G is a red-blue-edge-coloured complete graph with $|V(G)| \geq r(s,t)$, then G has either a red clique of size s or a blue clique of size t.

Moreover, if $s,t \geq 2$, we have

$$r(s,t) \leq r(s-1,t) + r(s,t-1).$$

Proof: Induction on (s,t), using the lexicographic ordering.

Base case: $r(s,1) = r(1,t) = 1$.

Now let $s > 1$ and $t > 1$.

Assume that the theorem holds for (s',t') such that

$$(1,1) \prec (s',t') \prec (s,t).$$

We want to show $r(s,t)$ exists, and

$$r(s,t) \leq r(s-1,t) + r(s,t-1).$$

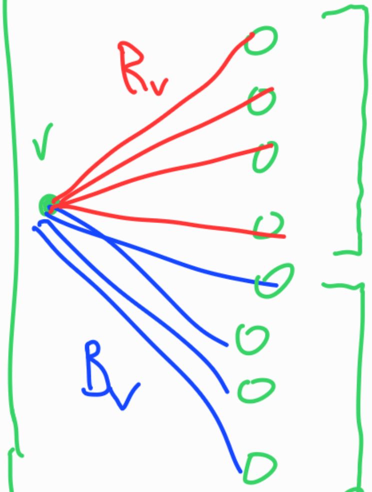
$s-1 \geq 1$ and $t-1 \geq 1$, so, by induction, the theorem holds for $(s-1,t)$ and $(s,t-1)$, i.e. $r(s-1,t)$ and $r(s,t-1)$ exist.

Let G be a red-blue edge-colored K_n with

$$n = r(s-1,t) + r(s,t-1).$$

Then, for $v \in V(G)$

$$d(v) = r(s-1,t) + r(s,t-1) - 1$$



We claim that there are either at least $r(s-1,t)$ of these vertices or at least $r(s,t-1)$ of these vertices.

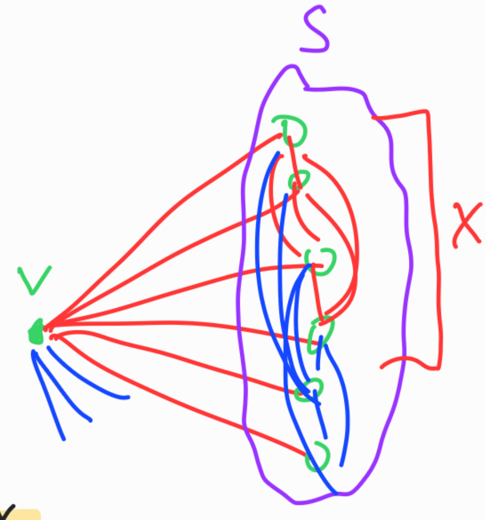
Let R_v be the red edges incident with v and let B_v be the blue edges incident with v.

If $|R_v| < r(s-1,t)$ and $|B_v| < r(s,t-1)$, then

$$d(v) = |R_v| + |B_v| \leq r(s-1,t) - 1 + r(s,t-1) - 1$$
$$= d(v) - 1,$$ a contradiction.

So either ① $|R_v| \geq r(s-1,t)$ or ② $|B_v| \geq r(s,t-1)$.

① Say $|R_v| \geq r(s-1,t)$.



Let S be the vertices joined to v by a red edge.

Then $|S| \geq r(s-1,t)$, so either $G[S]$ has a red clique X of size $s-1$, in which case $X \cup \{v\}$ is a red clique of size s in G, or $G[S]$, and hence G, has a blue clique of size t.

So the theorem holds for (s,t).

② Say $|B_v| \geq r(s,t-1)$.

Let S be the vertices joined to v by a blue edge.

Then either $G[S]$ has a red clique of size s, or $G[S]$ has a blue clique X of size $t-1$, in which case $X \cup \{v\}$ is a blue clique of size t in G.

The theorem follows by induction. □

We get Theorem 7.1 as a corollary of Thm 7.2.

For positive integers s, t, let $r(s,t)$ be the smallest positive integer such that any red-blue-edge-coloured complete graph on at least $r(s,t)$ vertices has either a red clique of size s or a blue clique of size t.

Also let $r(t) = r(t,t)$

Any such $r(s,t)$ is called a Ramsey number.

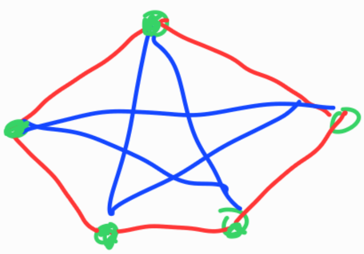
eg.

any 2-edge-colouring of K_6 has a monochromatic triangle.

So, the Ramsey number $r(3)$ is at most 6

i.e $r(3) \leq 6$.

Moreover, the following 2-edge-colouring of K_5 has no



monochromatic triangle, so $r(3) > 5$

Hence $r(3) = 6$.

We've seen $r(1,t)=1$ for all t.

<u>Lemma</u> 7.3: If $t \geq 2$, then $r(2,t) = t$.

$r(3,3)=6$ (above).

We know $r(4,4)=18$.

What about $r(5,5)$? Between 43 and 48.