MATH361 | Lecture 29

*course/teaching surveys on nuku

Recap: Hadwiger's Conjecture

Let G be a loopless k-chromatic graph for some positive integer k.

Then G has a K_k-minor.

True for $k \leq 6$. Open for $k \geq 7$.

Hajós made a similar conjecture back in 1950.

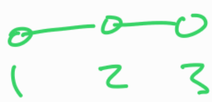
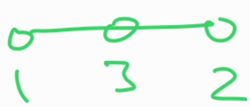
Conjecture 6.27:
Let G be a loopless k-chromatic graph for some positive integer k.

Then G has K_k as a topological minor.

True for $k \leq 4$, but disproved by Catlin for $k=8$ (1979).

Let G_n denote the number of simple graphs on the vertex set $\{1, 2, \ldots, n\}$.

Note: not up to isomorphism

e.g.  is different to .

Let P be a property that a graph may (or may not) have

e.g. having a Hamiltonian cycle

and let P_n be the number of simple graphs on the vertex set $\{1, 2, \ldots, n\}$ having property P.

A property P is rare if $\lim_{n \to \infty} \frac{P_n}{G_n} = 0$

A property P is typical if $\lim_{n \to \infty} \frac{P_n}{G_n} = 1$.

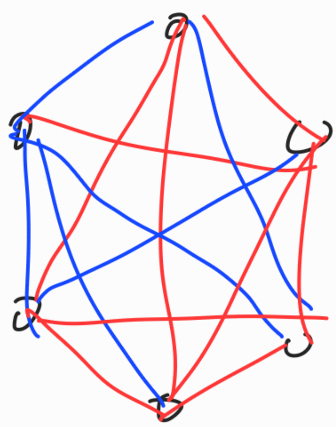
Theorem 6.29 (Erdős and Fajtlowicz, 1981)

The property of being a counterexample to Hajós's conjecture is typical.

Theorem 6.30 (Bollobás, 1980)

The property of being a counterexample to Hadwiger's conjecture is rare.

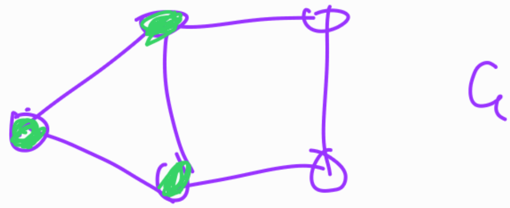
Ramsey Theory



In a graph G, a clique is a set $X \subseteq V(G)$ such that every pair of distinct vertices in X are adjacent.

For a simple graph, a clique is a set $X \subseteq V(G)$ such that $G[X] \cong K_{|X|}$.

e.g.



green vertices are a clique in G.

When X is a clique with $|X| = k$, then we say the clique has size k.

Consider a complete graph $G \approx K_n$. Then any subset $X \subseteq V(G)$ is a clique. Given a edge-colouring of G, we say a clique X is monochromatic if all the edges in $G[X]$ are the same colour.

Theorem 7.1 (Ramsey's Theorem, 1930)

For every positive integer t, there exists a number $r(t)$ such that if G is a 2-edge-colored complete graph with $|V(G)| \geq r(t)$, then G has a monochromatic clique of size t.

e.g. When $t=3$, let $r(3)=6$
and we get the case from the assignment question.

Theorem 7.2

For all positive integers s and t, there exists a number $r(s,t)$ such that if G is a red-blue edge-colored complete graph with $|V(G)| \geq r(s,t)$, then either G has a red clique of size s or a blue clique of size t.

Moreover, if $s,t \geq 2$, we have

$$r(s,t) \leq r(s-1,t) + r(s,t-1).$$

Observe that Theorem 7.1 follows from Theorem 7.2 by picking $r(t) = r(t,t)$ as given by Theorem 7.2.

Lexicographic ordering

Let S be a set with a relation $\prec$ on S.
Then $\prec$ is a linear ordering if it is asymmetric, transitive, irreflexive, and for all $a,b \in S$

with $a \neq b$, either $a \prec b$ or $b \prec a$.

(Sometimes also known as a strict total order).

A word is an ordered string $u_1 u_2 \dots u_s$

e.g. g r a p h

$u_1 = g$
$u_2 = r$
$\vdots$
$u_5 = h$

Let S be a set with a linear ordering.

Let S^* be a set of words using elements of S.

We define the lexicographic ordering on members of S^* as follows

Say $u = u_1 u_2 \dots u_s$ and $v = v_1 v_2 \dots v_t$ with $u \neq v$ and $s \leq t$

If $u_i = v_i$ for all $i \in \{1, 2, \dots, s\}$, then $u \prec v$.

Otherwise, let i be the smallest index such that $u_i \neq v_i$.

If $u_i \prec v_i$ then $u \prec v$.

Otherwise (when $v_i \prec u_i$), $v \prec u$.

e.g. using the usual linear order on the alphabet

limb $\prec$ limbic

calf $\prec$ can