MATH3611 | Lecture 18

Assignment 3 due 30 April

Recap:

- equivalent planar embeddings

- a planar graph G has a unique planar embedding if any two planar embeddings of G are equivalent

Theorem 5.3 (Whitney 1933)

Let G be a simple 3-connected planar graph. Then G has a unique planar embedding.

Corollary 5.4:

Let G be a simple 3-connected planar graph. Then G has a unique planar dual.

Euler's Formula

Let G be a plane graph.

We use $\left\{ \begin{array}{l} v(G) \\ e(G) \\ f(G) \end{array} \right\}$ to denote the number of $\left\{ \begin{array}{l} \text{vertices} \\ \text{edges} \\ \text{faces} \end{array} \right\}$ of G respectively.

Theorem 5.5 (Euler's formula):

Let G be a connected plane graph. Then

$$v(G) - e(G) + f(G) = 2$$

Proof: By induction on the number of edges.

If G has no edges, then $v(G)=1$, $f(G)=1$, so

$$1 - 0 + 1 = 2 \quad \text{as required.}$$

Now assume G has at least one edge, and that Euler's formula holds for any graph with fewer than $e(G)$ edges.

Let e be an edge of G. Suppose e is not a loop.

We've seen Lemma 4.11, that says when e is not a bridge, each face boundary of G/e is either

- a face boundary of G (when e is not incident to the face)
- a face boundary of G but with e removed. (when e is incident to the face)

Therefore $f(G/e) = f(G)$. This is also the case when e is a bridge.

Since e is not a loop, $v(G/e) = v(G) - 1$.

And $e(G/e) = e(G) - 1$.

By the induction assumption

$$v(G/e) - e(G/e) + f(G/e) = 2, \text{ so}$$

$$(v(G) \cancel{-1}) - (e(G) \cancel{-1}) + f(G) = 2$$

implying $v(G) - e(G) + f(G) = 2$ as req^{d}.

We still need to consider when e is a loop:

In this case $e(G/e) = e(G) - 1$

$$v(G/e) = v(G)$$

$$f(G/e) = f(G) - 1$$

Again by the induction assumption:

$$v(G/e) - e(G/e) + f(G/e) = 2, \text{ so}$$

so $v(G) - (e(G) - 1) + (f(G) - 1) = 2$,

implying $v(G) - e(G) + f(G) = 2$ as req^d. $\square$

Euler's formula has useful corollaries. Namely:

<u>Corollary 5.6</u>: For a connected plane graph G, every planar embedding of G has the same number of faces.

<u>Corollary 5.7</u> Let G be a simple planar graph with at least 3 vertices. Then $e(G) \leq 3v(G) - 6$.

Corollary 5.7 tells us that for a simple planar graph, the maximum number of edges grows (at most) linearly

in the number of vertices.

Recall K_n has $\frac{n(n-1)}{2}$ edges

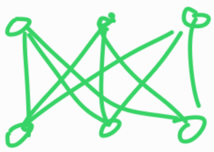
(i.e a quadratic number of edges relative to the number of vertices).

When n is large

$$\frac{n(n-1)}{2} >> 3n-6 .$$

If we consider all simple graphs on n vertices for large n, most are not planar.

Note: Corollary 5.7 is not a characterisation of planar graphs.



e.g. For $K_{3,3}$, $e(K_{3,3}) = 9$

$$v(K_{3,3}) = 6$$

So, as $9 \leq 3 \times 6 - 6$, Corollary 5.7 tells us nothing about whether or not $K_{3,3}$ is planar.

Wagner's Theorem

Recall: for graphs G and H

We say **G has an H-minor**

when there is a minor H' of G such that H' is isomorphic to H

1) We've seen K_5 and $K_{3,3}$ are not planar. (Thm 4.3, Ex 4.4)

2) If G is planar, any minor of G is also planar (Corollary 4.12).

Thus:

If G has a K_5-minor or $K_{3,3}$-minor then G is not planar.

Remarkably, the converse is also true:

Theorem 5.10 (Wagner 1937; Kuratowski 1930)

A graph is planar if and only if it has no K_5-minor and no $K_{3,3}$-minor

Some consequences:

- to show that a graph is not planar, now we just need to show it has a K_5- or $K_{3,3}$-minor.

- in fact, there is an efficient algorithm to determine whether or not a graph has a K_5- or $K_{3,3}$-minor.