MATH361 | Lecture 17

Assignment 3 due 30 April 3pm

For a connected plane graph G

$(G \backslash e)^* = G^* / e^*$ when e is not a bridge

$(G/e)^* = G^* \backslash e^*$ when e is not a loop

(Lemmas 4.19 + 4.20).

If C is a cycle of G, then C^* is a bond of G^*

If B is a bond of G, then B^* is a cycle of G^*

(Th^m 4.24)

If C and D are $\begin{cases} \text{cycles} \\ \text{bonds} \end{cases}$ and $e \in C \cap D$, then $(C \cup D) \backslash \{e\}$ contains a $\begin{cases} \text{cycle} \\ \text{bond} \end{cases}$

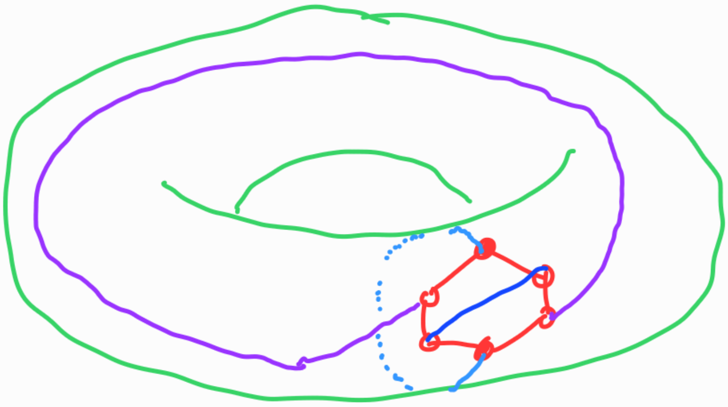
(4.25, 4.26)

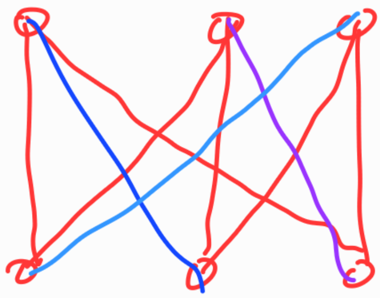
Lemmas 4.19/4.20 can also be used to prove the following:

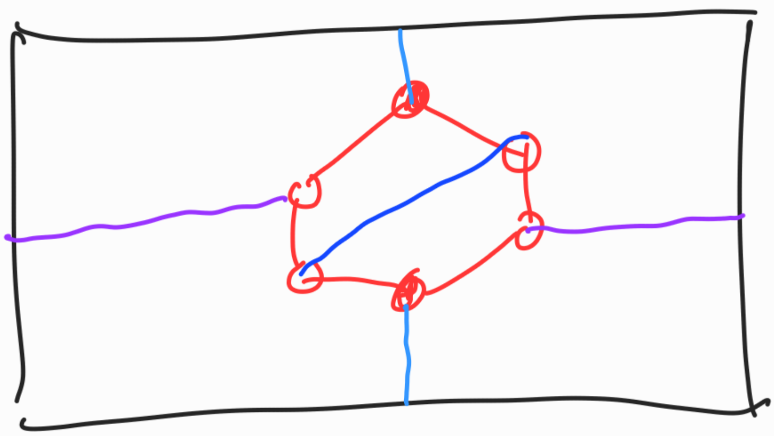
Thm 4.21: Let G be a loopless 2-connected plane graph with at least 3 faces. Then G^* is loopless and 2-connected.

So far, we're focussed on embedding a graph in the plane, but we could just as well consider embedding a graph in any surface.

$K_{3,3}$ can be embedded on the torus with no edges crossing:







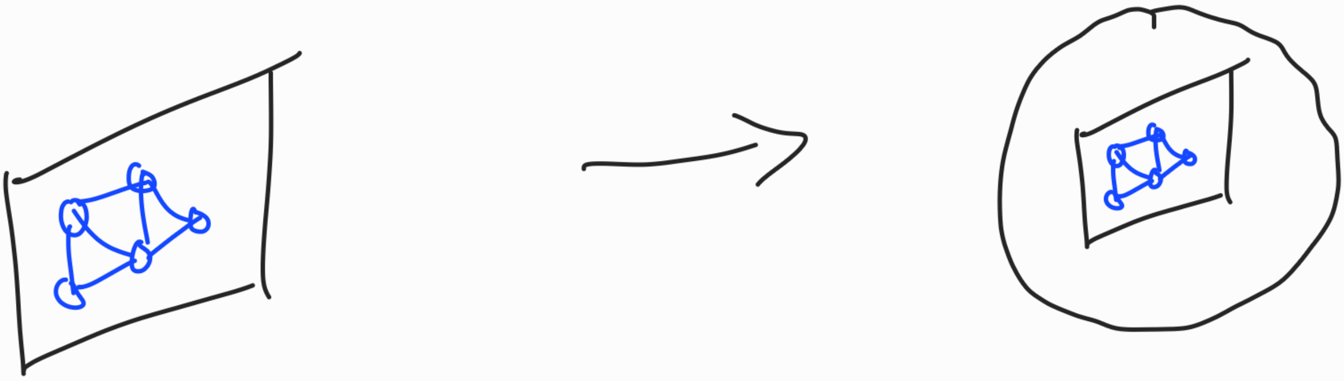
← this is a simpler way to illustrate this embedding.

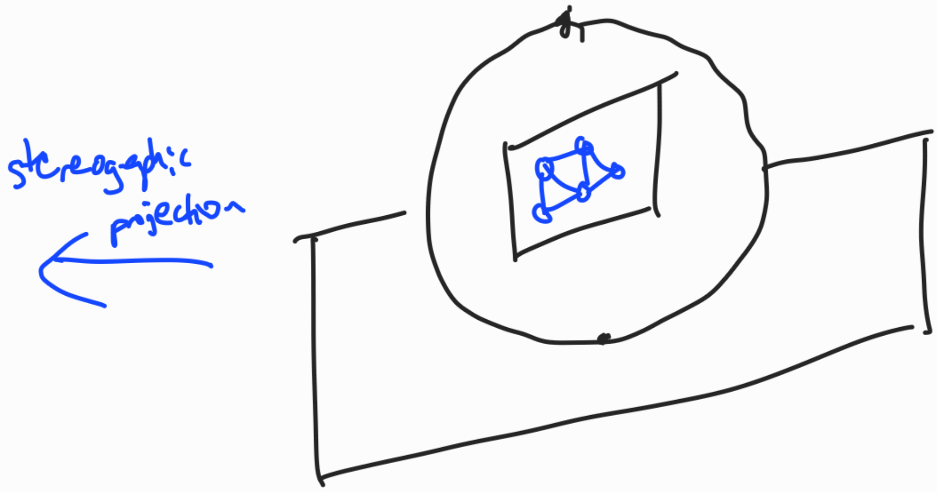
One other surface of interest is the surface of a 3-dimensional sphere.

<u>Theorem 4.7</u>: A graph can be embedded on the plane (with no edges crossing) if and only if it can

be embedded on the sphere (with no edges crossing)

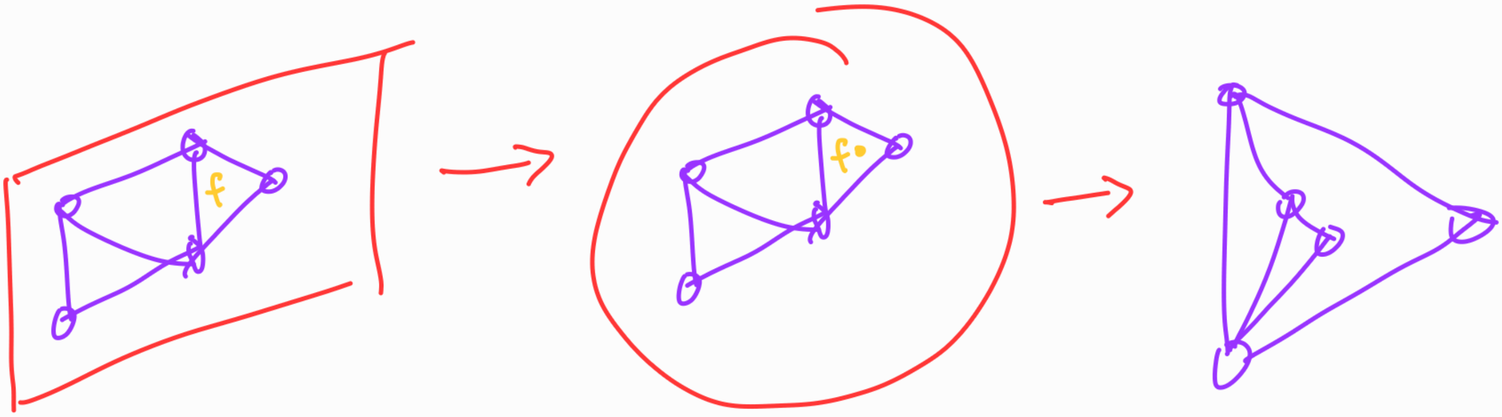
Sketch proof:





Note: for an embedding on the plane, there is a distinguished outer face — unlike an embedding on the sphere.

Theorem 4.8: Let G be a planar graph, and let f be a face in a plane graph that embeds G. Then there exists a planar embedding of G for which the outer face has the same boundary as f.



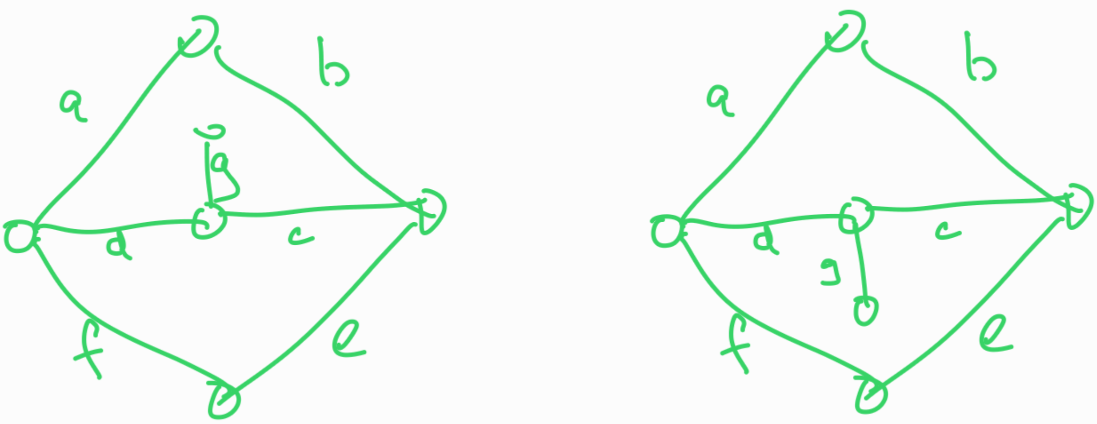
Planar graphs II

Recall: Different embeddings of the same planar graph can have different planar duals.

Q: Given two planar embeddings of a graph, do they have the same face boundaries?

A: No

e.g. 1

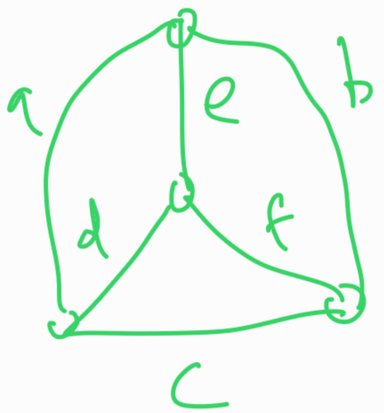


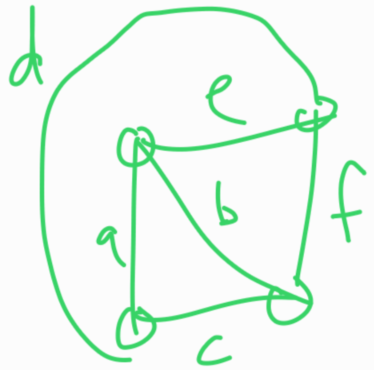
(inequivalent planar embeddings)

These are two different planar embeddings of a graph G, but the left has a face boundary with edges $\{c, e, f, d\}$ but the right does not.

We say that two embeddings of a planar graph are equivalent if they have the same set of face boundaries; otherwise they are inequivalent.

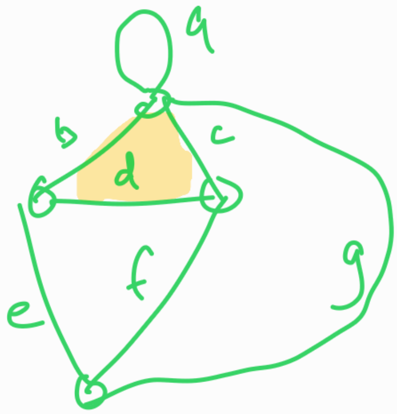
e.g. 2 (equivalent planar embeddings)

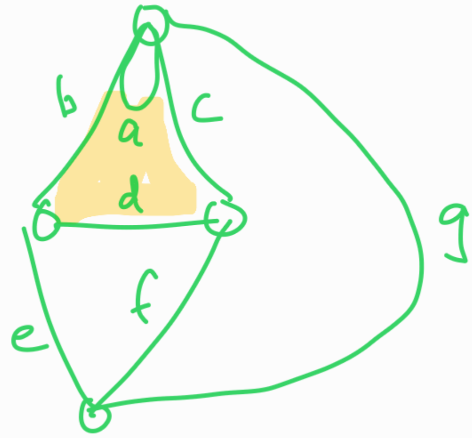




These two embeddings of K_4 are equivalent as they have the same set of face boundaries.

e.g. 3





Here, the underlying graph is 3-connected but these planar embeddings are inequivalent.

We say that a planar graph G has a unique planar embedding if any two planar embeddings of G are equivalent.

Theorem 5.3 (Whitney 1933)

Let G be a simple 3-connected planar graph. Then G has a unique planar embedding.