# MATH361 Lecture 16

Previously, we saw

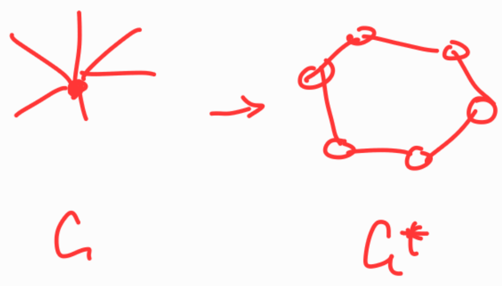
→ plane graphs

→ the planar dual $G^*$ of a plane graph $G$

$G^*$ is connected (Lemma 4.16)

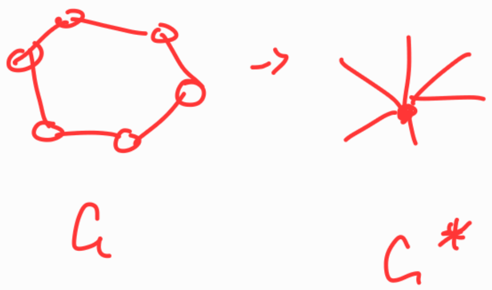
For a connected plane graph:

the edges incident with a vertex in $G$ correspond to edges incident with a face of $G^*$




the edges incident with a face in $G$ correspond to edges incident with a vertex of $G^*$

(Lemma 4.17).




and $G^{**} = G$.

**Lemma:** If $e$ is a bridge in a plane graph $G$, then $e^*$ is a loop in $G^*$.

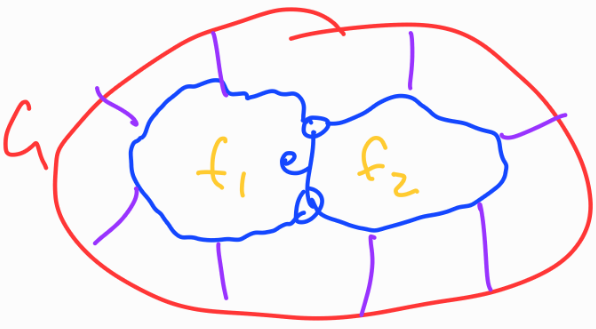
**Proof:** Since $e$ is a bridge, it is only incident to one face $f$ (by Lemma 4.9 (iii)). Thus the edge $e^*$ is incident only to $f^*$, i.e $e^*$ is a loop.

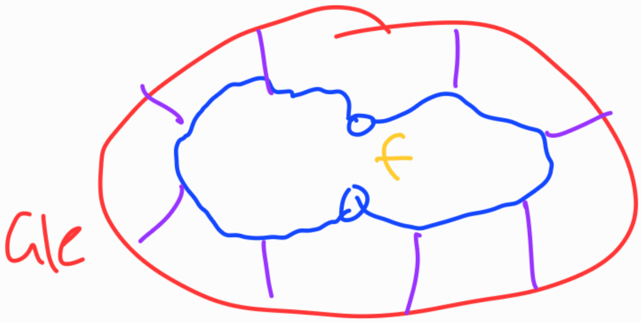
**Lemma 4.19:** Let $G$ be a connected plane graph, and let $e$ be an edge of $G$ that is not a bridge. Then

$$(G\backslash e)^* = G^*/e^*$$

$$\begin{tikzcd} & G\backslash e \arrow[dr, "*"] & \\ G \arrow[ur, "\backslash e"] \arrow[dr, "*"'] & & (G\backslash e)^* = G^*/e^* \\ & G^* \arrow[ur, "/e^*"'] & \end{tikzcd}$$

**Proof:** As $e$ is not a bridge, it is incident to 2 faces, $f_1$ and $f_2$ say. When we delete $e$, the faces $f_1$ and $f_2$ become a single face, $f$ say.




Otherwise, a face $g \in F(G) \setminus \{f_1, f_2\}$ is also a face in $G\backslash e$. Moreover, $g$ is adjacent to $f$ in $G\backslash e$ iff it is adjacent to $f_1$ or $f_2$ in $G$ — otherwise, adjacencies don't change.

Therefore in $(G \setminus e)^*$, the vertices $f_1^*$ and $f_2^*$ become a single vertex $f^*$, otherwise the vertices are the same as in $G^*$.

A vertex $g^*$ is adjacent to $f^*$ in $(G \setminus e)^*$ iff it is adjacent to $f_1^*$ or $f_2^*$ in $G^*$ — otherwise adjacencies don't change.

This is precisely the definition of the contraction of $e^*$ from $G^*$. $\square$

**Lemma 4.20** Let $G$ be a connected plane graph, and let $e$ be an edge of $G$ that is not a loop.

Then

$$(G/e)^* = G^* \setminus e^*$$

> **Lemma 4.19**: Let $G$ be a connected plane graph, and let $e$ be an edge of $G$ that is not a bridge.
>
> Then
>
> $$(G \setminus e)^* = G^*/e^*$$

**Proof**: $G^*$ is a connected plane graph (by Lemma 4.16) and $e^*$ is not a bridge (since if $e^*$ were a bridge, then $e^{**} = e$ is a loop in $G^{**} = G$, by the earlier claim).

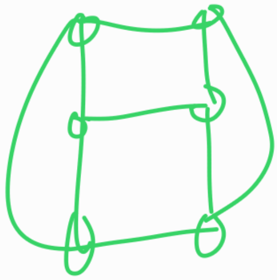
So

$$(G^*\backslash e^*)^* = G^{**}/e \quad \text{by Lemma 4.19,}$$
$$= G/e \quad \text{by Thm 4.18}$$

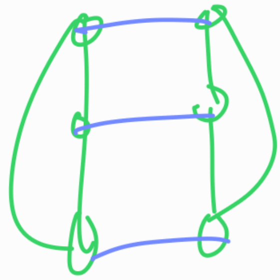
Taking duals of both sides
$$G^*\backslash e^* = (G/e)^* \quad \text{as required.} \quad \square$$

A **bond** in a connected graph is a minimal set of edges whose removal disconnects the graph.

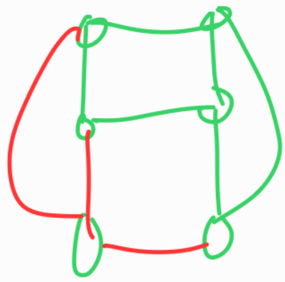
e.g.



G



the blue edges are a bond.



the red edges are a bond of G

For the purposes of this lecture, we view a cycle as a set of edges (rather than a subgraph).

For a plane graph $G$, and a set $X \subseteq E(G)$

let $X^* = \{e^* : e \in X\}$.

<u>Theorem</u> 4.22: Let $G$ be a connected plane graph.

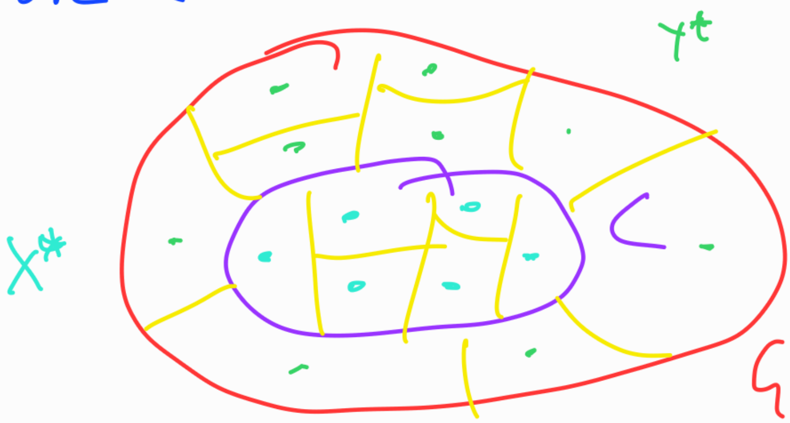
i) If $C$ is a cycle of $G$, then $C^*$ is a bond of $G^*$

ii) If $B$ is a bond of $G$, then $B^*$ is a cycle of $G^*$.

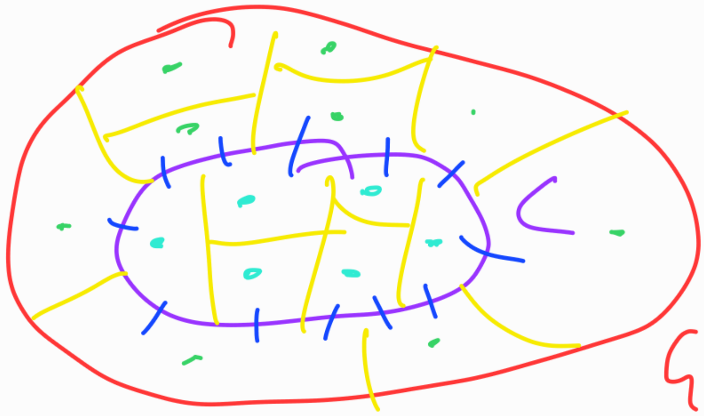
<u>Lemma 4.21</u>: Let $G$ be a plane graph, let $C$ be a cycle of $G$, and let

$$\left.\begin{matrix} X^* \\ Y^* \end{matrix}\right\} \text{ be the vertices of } G^* \text{ in the } \left\{\begin{matrix} \text{interior} \\ \text{exterior} \end{matrix}\right\} \text{ of } C.$$

$G^*[X^*]$ and $G^*[Y^*]$ are connected.



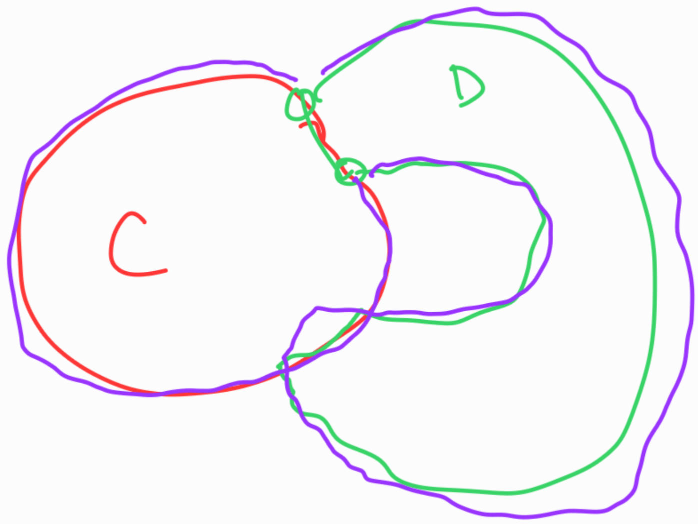

To prove Thm 4.22(i)...




Observe that every $(x^*, y^*)$-path in $G^*$ contains an edge in $C^*$, so $G \setminus C^*$ is not connected. It remains to show that $C^*$ is minimal with respect to this property (use Lemma 4.21).

**Lemma 4.23:** Let $G$ be a graph, and let $C$ and $D$ be cycles of $G$, with $e \in C \cap D$. Then $(C \cup D) \setminus \{e\}$ contains a cycle.

Proof idea:




closed walk on $(C \cup D) \setminus \{e\}$ contains a cycle.

**Corollary 4.24:** Let $G$ be a connected plane graph

Let $C$ and $D$ be bonds of $G$ with $e \in C \cap D$.

Then $(C \cup D) \setminus \{e\}$ contains a bond.