MATH361 | Lecture 10

Assignment 2 due 26 March

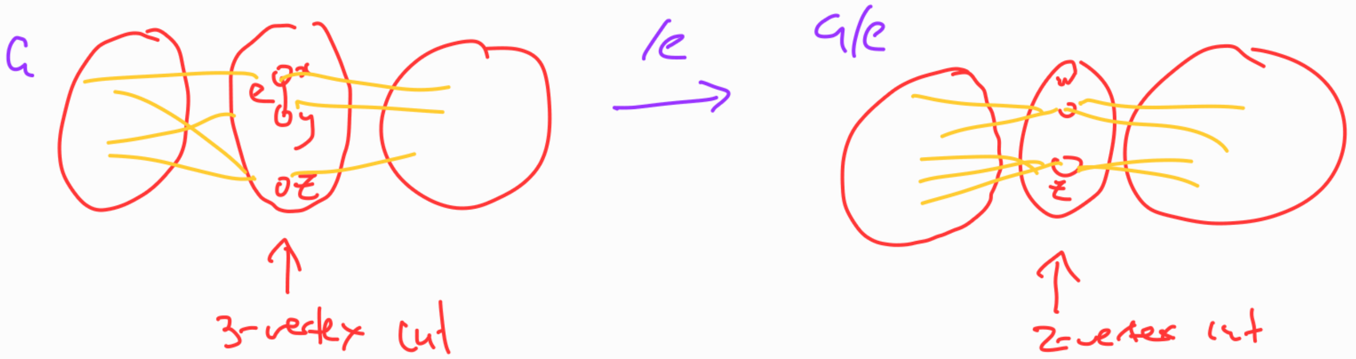
2^{nd} half of section 3 notes online

Last time: 2-connected graphs + blocks

3-connected graphs $\rightarrow$ connected, ≥ 4 vertices

no 1- or 2-vertex cuts

Theorem 3.21: Let G be a 3-connected graph with $|V(G)| \geq 5$. Then there exists $e \in E(G)$ such that G/e is 3-connected.



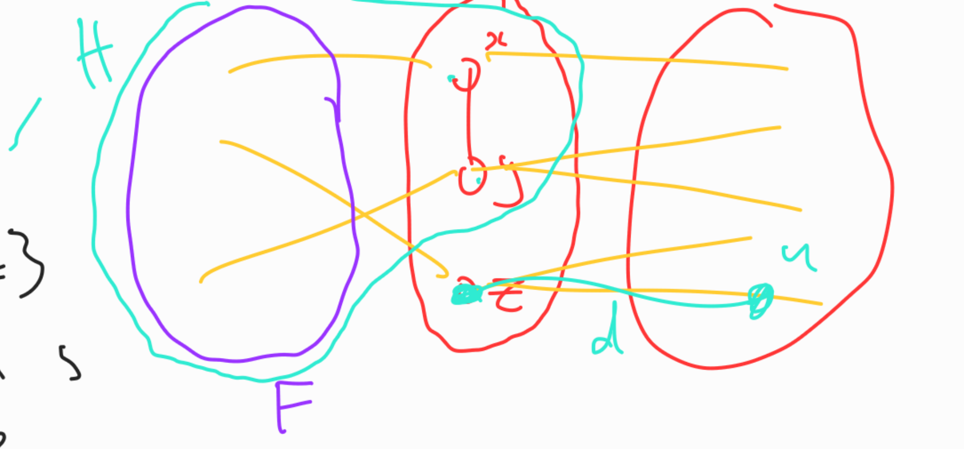
Lemma 3.22: Let G be a 3-connected, with $|V(G)| \geq 5$. If G/e is not 3-connected, for some $e = xy$, then there exists a vertex z such that $\{x, y, z\}$ is a 3-vertex cut of G.

Proof left as an exercise.

Proof of Thm 3.21: Towards a contradiction, suppose G has no edge e such that G/e is 3-connected.

Then for every $e \in E(G)$ with ends x and y, there exists a vertex z such that $\{x, y, z\}$ is a 3-vertex cut, by Lemma 3.22.

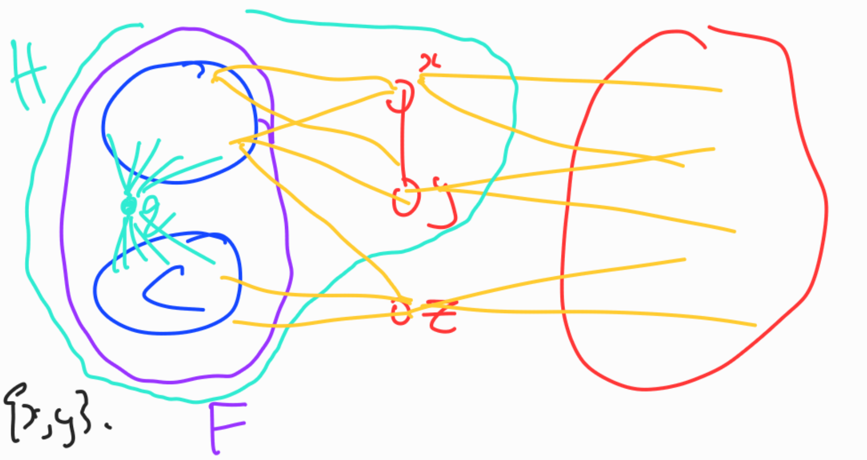
Amongst all choices for $e = xy$ and z, choose these so that $G - \{x, y, z\}$ has a component F with as many vertices as possible



Let $H = G[V(F) \cup \{x, y\}]$. Claim: H is 2-connected

Suppose H has a cut vertex q. Then $H - q$ has more than 1 component. Note that $q \notin \{x, y\}$ for otherwise $G - \{q, z\}$ is disconnected, contradicting that G is 3-connected.

Since $e = xy$ is an edge, $\{x, y\}$ is contained in a component of $H - q$. Let C be the vertex set of a component of $H - q$ not containing $\{x, y\}$.



Now every path from a vertex in C to a vertex in $V(G) \setminus (C \cup \{q, z\})$ passes through q or z in G. So $\{q, z\}$ is a 2-vertex cut in G, contradicting that G is 3-connected. From this contradiction, we deduce H is 2-connected.

Let $d = zu$ be an edge joining z to a vertex not in H (such an edge exists since otherwise $\{x,y\}$ is a contradicting 2-vertex cut in G).

Then G/d is not 3-connected, so by Lemma 3.22 again, there is a vertex v such that $\{z,u,v\}$ is a 3-vertex cut.

Let $H' = \begin{cases} H & \text{if } v \notin V(H) \\ H-v & \text{if } v \in V(H). \end{cases}$

Then H' is connected, since H is 2-connected.

So H' is contained in a component of $G-\{z,u,v\}$.

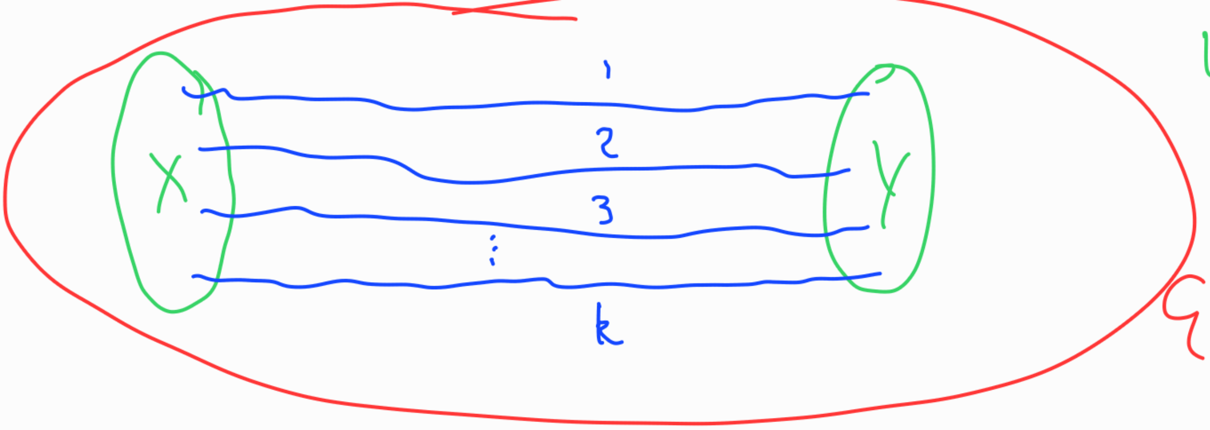
But $|V(H')| \geq |V(H)|-1 \geq |V(F)|+1$, a contradiction to the fact that F was chosen to have maximum size. $\square$

<u>Lemma</u> 3.18: If G is k-connected, for $k \geq 2$, and $v \in V(G)$, then $G-v$ is $(k-1)$-connected.

<u>Lemma</u> 3.19: If G is k-connected, for $k \geq 2$, and $e \in E(G)$, then $G \setminus e$ is $(k-1)$-connected.

<u>Menger's theorem</u>

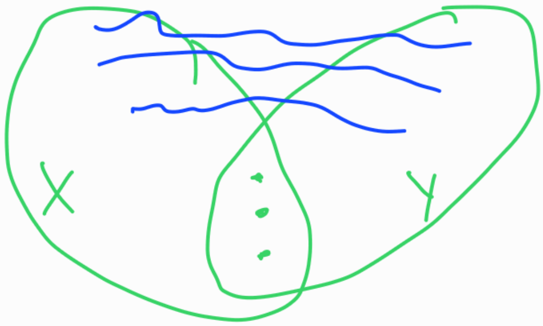
Suppose we have two sets of vertices X and Y in a graph G.



Let k be a positive integer.

When can we find k disjoint paths from a vertex in X to a vertex in Y.

Note: we need not assume X and Y are disjoint, but any vertex in $X \cap Y$ is a trivial path from X to Y.



We need some terminology:

* for $X, Y \subseteq V(G)$, an (X,Y)-path is a path from a vertex in X to a vertex in Y.

* for $x, y \in V(G)$, an (x,y)-path is a path from x to y.

* for a path P of G, let $V(P)$ denote the vertices of P

and $E(P)$ —— edges of P.

* for paths P and Q, the paths P and Q
are vertex-disjoint if $V(P) \cap V(Q) = \emptyset$
and edge-disjoint if $E(P) \cap E(Q) = \emptyset$.