Victoria University of Wellington
School of Mathematics and Statistics

MATH 361	Assignment 4	T1 2024

Due 3pm Tuesday 14 May

1. Recall that Euler's formula (as we saw in lectures as Theorem 5.5) concerns *connected* plane graphs.

(a) Give an example to demonstrate that Euler's formula does not hold if G is not connected.
(b) Let G be a plane graph, and let $c(G)$ be the number of components of G. State a generalisation of Euler's Formula that describes the relationship between $v(G)$, $e(G)$, $f(G)$ and $c(G)$. Prove this generalisation.

2. Consider the following graph classes:

(a) The class of graphs with at most one cycle.
(b) The class of *pseudotrees*, consisting of connected graphs with at most one cycle.
(c) The class of *pseudoforests*, consisting of graphs where each component has at most one cycle.

For each of graph classes (a)–(c), answer the following questions:

(i) Is the class minor-closed?
(ii) If yes, what are the excluded minors? If no, give an explicit example to demonstrate this.

3. Say that for any pair of people, they either both know each other, or they are strangers (neither knows each other). Prove that in any party of six people, there is a group of three that either all know each other, or all are strangers.

(Hint: how can you model this problem using a graph?)

4. Before attempting this question, see Tutorial Q6. As in that question, let $\mathcal{H}$ be the smallest class that contains the graphs K_5' and $K_{3,3}'$, and is closed under isomorphism and 2-sum. Prove that every graph in $\mathcal{H}$ is planar.

5. Show that K_6 is ΔY equivalent to the Petersen graph.

Tutorial exercises:

1. Prove Corollary 5.6: "All planar embeddings of a connected planar graph have the same number of faces."

2. Prove Corollary 5.7: "Let G be a simple planar graph with at least three vertices. Then $e(G) \leq 3v(G) - 6$."

3. Consider the following graph classes:

(a) The class of *path graphs*, consisting of graphs isomorphic to P_n for some non-negative integer n. (Note that the empty graph is in this class.)
(b) The class of *linear forests*, consisting of graphs where each component is isomorphic to a path graph.
(c) The class of *unicyclic graphs*, consisting of graphs with exactly one cycle.
(d) The class of *cactus graphs*, consisting of graphs where any two cycles share at most one vertex.

For each of graph classes (a)–(d), answer the following questions:

(i) Is the class minor-closed?
(ii) If yes, what are the excluded minors? If no, give an explicit example to demonstrate this.

4. Exercise 5.18: Give an example of graphs G and H such that G has H as a minor, but H is not a topological minor of G.

5. Exercise 5.24: Prove that K_4 is $Y\Delta Y$ reducible.

6. Recall that we defined the *union* of two consistent graphs G_1 and G_2 in Lecture 9, and we denoted this $G_1 \cup G_2$.

Let G and H be graphs with a single edge e in common; that is, when e has ends u and v, we have $E(G) \cap E(H) = \{e\}$ and $V(G) \cap V(H) = \{u, v\}$. Then G and H are consistent. We define the 2-*sum* of G and H, denoted $G \oplus_2 H$, to be the graph $(G \cup H) \backslash e$. Loosely speaking, $G \oplus_2 H$ is the graph obtained by identifying G and H on the edge e, then deleting e.

(a) Let K_5' be a graph obtained by deleting an edge of K_5, and let $K_{3,3}'$ be a graph obtained by deleting an edge of $K_{3,3}$, where K_5' and $K_{3,3}'$ have a single edge e in common. Provide planar drawings of four non-isomorphic graphs that can be obtained as the 2-sum of a graph isomorphic to K_5', and a graph isomorphic to $K_{3,3}'$.

(b) Recall that a class of graphs $\mathcal{G}$ is *closed under isomorphism* if whenever G is in $\mathcal{G}$, then any graph that is isomorphic to G is also in $\mathcal{G}$. We say that a class of graphs $\mathcal{G}$ is *closed under 2-sum* if for any two graphs in $\mathcal{G}$ with a single edge in common, the 2-sum of these graphs is also in $\mathcal{G}$. We define $\mathcal{H}$ to be the smallest class of graphs that contains K_5' and $K_{3,3}'$, and is closed under isomorphism and 2-sum. Prove that every graph in $\mathcal{H}$ is 2-connected.