Victoria University of Wellington

School of Mathematics and Statistics

MATH 361	Assignment 3 Solutions	T1 2024

Due 3pm Tuesday 30 April

1. For a graph G with vertex set V, and disjoint sets $A, B \subseteq V$, let $e(A, B)$ be the number of edges of G with one end in A and the other end in B, and let $d(A) = e(A, V \setminus A)$. Prove, for any $X, Y \subseteq V$, that

$$d(X) + d(Y) \geq d(X \cup Y) + d(X \cap Y).$$

(Hint: one option is to show that $d(X)+d(Y) = d(X\cup Y)+d(X\cap Y)+2e(X\setminus Y, Y\setminus X)$.)

Solution. There are different ways to approach this question; my approach may not be the most elegant, but I'll try to make the working as clear as I can. First, observe that for any $A \subseteq V(G)$, we have $e(A, V \setminus A) = e(V \setminus A, A)$, so $d(A) = d(V \setminus A)$. We'll use this frequently below.

To simplify notation, we let $\overline{X} = V \setminus X$ and $\overline{Y} = V \setminus Y$. Then, every vertex in V belongs to one of the four disjoint sets $X \cap Y$, $X \cap \overline{Y}$, $\overline{X} \cap Y$, or $\overline{X} \cap \overline{Y}$ (as the union of these four sets is V). First consider $d(X \cap Y)$. If an edge has one end in $X \cap Y$ and the other end in $V \setminus (X \cap Y)$, then this other end is in one of the three disjoint sets $X \cap \overline{Y}$, $\overline{X} \cap Y$, or $\overline{X} \cap \overline{Y}$ (as the union of these three sets is $V \setminus (X \cap Y)$. So

$$d(X \cap Y) = e(X \cap Y, X \cap \overline{Y}) + e(X \cap Y, \overline{X} \cap Y) + e(X \cap Y, \overline{X} \cap \overline{Y}).$$

Note that $d(X \cup Y) = e(X \cup Y, \overline{X} \cap \overline{Y}) = d(\overline{X} \cap \overline{Y})$. So, we also have

$$\begin{aligned} d(X \cup Y) &= d(\overline{X} \cap \overline{Y}) \\ &= e(\overline{X} \cap Y, \overline{X} \cap \overline{Y}) + e(X \cap \overline{Y}, \overline{X} \cap \overline{Y}) + e(X \cap Y, \overline{X} \cap \overline{Y}). \end{aligned}$$

Next consider $d(X)$. We have that X is the disjoint union of $X \cap Y$ and $X \cap \overline{Y}$, and $\overline{X}$ is the disjoint union of $\overline{X} \cap Y$ and $\overline{X} \cap \overline{Y}$. Therefore

$$d(X) = e(X \cap Y, \overline{X} \cap Y) + e(X \cap Y, \overline{X} \cap \overline{Y}) + e(X \cap \overline{Y}, \overline{X} \cap \overline{Y}) + e(X \cap \overline{Y}, \overline{X} \cap Y).$$

Similarly

$$d(Y) = e(X \cap Y, X \cap \overline{Y}) + e(X \cap Y, \overline{X} \cap \overline{Y}) + e(\overline{X} \cap Y, \overline{X} \cap \overline{Y}) + e(\overline{X} \cap Y, X \cap \overline{Y}).$$

Now we put it all together. We have

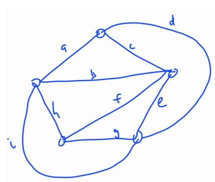
$$\begin{aligned}
d(X)+d(Y) &= e(X\cap Y,\overline{X}\cap Y)+e(X\cap Y,\overline{X}\cap\overline{Y})+e(X\cap\overline{Y},\overline{X}\cap\overline{Y})+e(X\cap\overline{Y},\overline{X}\cap Y)\\
&\quad+e(X\cap Y,X\cap\overline{Y})+e(X\cap Y,\overline{X}\cap\overline{Y})+e(\overline{X}\cap Y,\overline{X}\cap\overline{Y})+e(\overline{X}\cap Y,X\cap\overline{Y})\\
&= e(X\cap Y,X\cap\overline{Y})+e(X\cap Y,\overline{X}\cap Y)+e(X\cap Y,\overline{X}\cap\overline{Y})\\
&\quad+e(\overline{X}\cap Y,\overline{X}\cap\overline{Y})+e(X\cap\overline{Y},\overline{X}\cap\overline{Y})+e(X\cap Y,\overline{X}\cap\overline{Y})\\
&\quad+e(X\cap\overline{Y},\overline{X}\cap Y)+e(\overline{X}\cap Y,X\cap\overline{Y})\\
&= d(X\cap Y)+d(X\cup Y)+2e(X\setminus Y,Y\setminus X).
\end{aligned}$$

Now, as $e(X\setminus Y,Y\setminus X)\geq 0$, we have that $d(X)+d(Y)\geq d(X\cap Y)+d(X\cup Y)$, as required. □

2. Let K_5' be the graph obtained by deleting an edge from K_5.

(a) Draw a planar embedding of K_5'. Clearly label the edges.

Solution. See below:



(b) How many faces does your embedding have? For each face, list the edges in the boundary of that face.

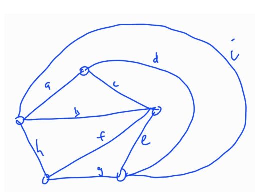
Solution. There are six faces. For my drawing, these faces have the following edges in each boundary (of course, for the latter part these will vary based on the answer to part (a)).

$$a,b,c;\quad c,d,e;\quad e,f,g;\quad g,h,i;\quad b,f,h;\quad a,d,i.$$

The outer face has the edges a,d,i in its boundary.

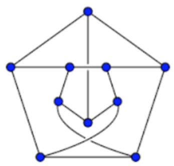
(c) Now choose a face that is *not* the outer face and draw another planar embedding of G in which that face is the outer face.

Solution. I chose the face with the edges g,h,i in its boundary. See picture below:



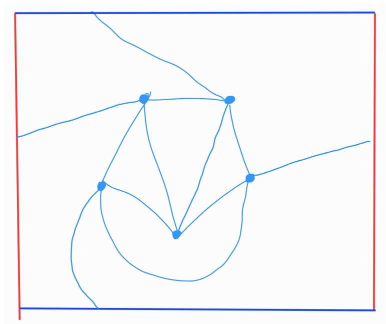
3. The Petersen Graph is illustrated in Tutorial Q3, with five edge crossings. Find a drawing of the Petersen Graph on the plane with only two edge crossings.

Solution. See picture below:



4. Show that K_5 can be embedded on the torus. (Hint: see Tutorial Q4 before attempting this question.)

Solution. See picture below:



5. (Exercise 4.4 or Corollary 5.8) By any method, prove that K_5 is not planar.

Solution. K_5 has 5 vertices and 10 edges. But $10 > 3 \times 5 - 6$. This violates the inequality of Corollary 5.7. Hence K_5 is not planar. □

6. (Exercise 5.9) Prove that $K_{3,3}$ is not planar using Euler's formula. (Hint: use together with the Handshaking Lemma for faces.)

Solution. Suppose that $K_{3,3}$ is planar and let G be a planar embedding of $K_{3,3}$. Because $K_{3,3}$ has no cycle of length less than four, every face of G has degree at least four. Hence, by the Handshaking Lemma for faces, we have

$$4f(G) \leq \sum_{f \in F(G)} d(f) = 2e(G) = 18$$

implying that $f(G) \leq 4$. By Euler's formula we have

$$2 = v(G) - e(G) + f(G) \leq 6 - 9 + 4 = 1.$$

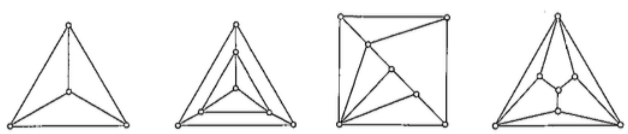
It follows from this contradiction that $K_{3,3}$ is not planar. □

7. A plane graph is *self-dual* if it is isomorphic to its dual.

(a) Show that if G is self dual, then $|E(G)| = 2|V(G)| - 2$.

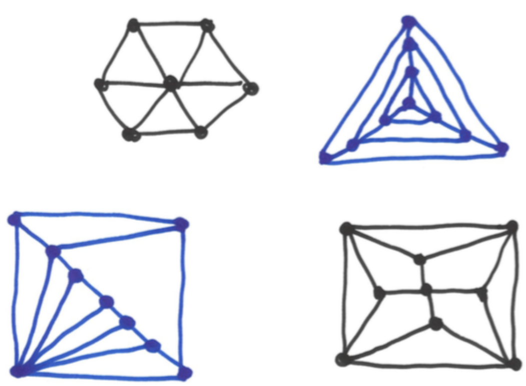
Solution. As G is self dual, and the vertices of G^* are the faces of G, we must have $|V(G)| = |V(G^*)| = |F(G)|$. By Euler's Formula $|F(G)| + |V(G)| - |E(G)| = 2$. Hence $2|V(G)| - |E(G)| = 2$ so that $|E(G)| = 2|V(G)| - 2$. □

(b) The four plane graphs illustrated below are self-dual. Each belongs to an infinite family of self-dual plane graphs. Describe one of these infinite families.



Solution. The first graph is the wheel on four vertices, and it is not too difficult to see that larger wheels are also self-dual. So the wheel graphs are one such infinite family.

The others are more difficult to describe. However, the figure below illustrates a further graph from each of the four classes. You should be able to deduce the pattern from this (even if it remains difficult to describe formally).



□

Smith and Tutte (1950) proved that members of these classes were the only 3-connected self-dual planar graphs.