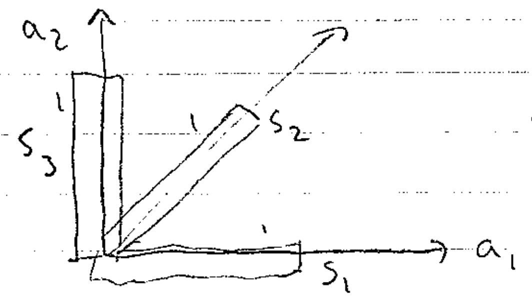
Tutorial Four Notes - for Assignment 3

(1) 3 strain Gauges measure the change in length of wires or rods, etc., deployed 45° apart in a plane as shown:



original lengths are $S_1 = S_2 = S_3 = 1$

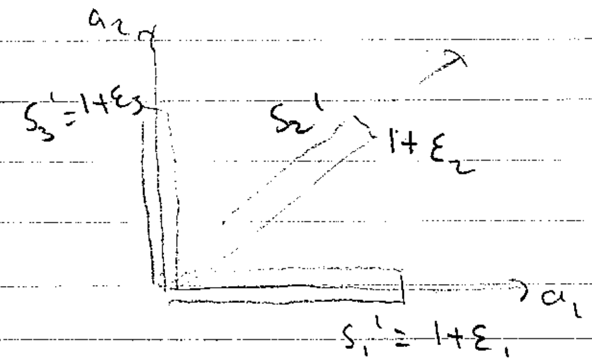
Assume $\hat{a}_3$ has no change (into + out of paper)

New lengths: have had changes

ε_1 per unit length in direction of S_1, which is a_1

ε_2 " " " " " " S_2, which is 45° to a_1

ε_3 " " " " " " S_3, which is a_2



Calculate Strain Tensor E using the strain gauge measurements

Remember formula:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i}\right)$$ in the $\hat{a}_1, \hat{a}_2, \hat{a}_3$ system.

$S_1' = \hat{a}_1(1+\varepsilon_1)$ $\qquad \Delta S_1 = S_1' - S_1 = \varepsilon_1 \hat{a}_1$

$S_2' = (1+\varepsilon_2)\cos\theta\, \hat{a}_1 + (1+\varepsilon_3)\sin\theta\, \hat{a}_2$ $\qquad \Delta S_2 = \varepsilon_2 \cos\theta\, \hat{a}_1 + \varepsilon_3 \sin\theta\, \hat{a}_2$

$S_3' = \hat{a}_2(1+\varepsilon_3)$ $\qquad \Delta S_3 = \varepsilon_3 \hat{a}_3$

$$E = \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}$$

Consider the unit vector $\hat{a}_1 : \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\Delta S_1 = \Delta u_1 = E\,\hat{a}_1 = E\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} E_{11} \\ E_{12} \end{pmatrix}$$

The component of this in the direction of the strain gauge is dot product:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} E_{11} \\ E_{12} \end{pmatrix} = E_{11} = \varepsilon_1 \quad \text{as defined.}$$

Similarly the component in direction $\hat{a}_2$ of the strain gauge is

$$\Delta S_2 = \Delta u_2 = E\,\hat{a}_2 = E\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} E_{12} \\ E_{22} \end{pmatrix}$$

Component of strain in direction $\hat{a}_2$ is

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} E_{12} \\ E_{22} \end{pmatrix} = E_{22} = \varepsilon_3 \quad \text{as defined}$$

Similarly for $S^{21} \rightarrow$

$$\begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & E_{12} \\ E_{12} & \varepsilon_3 \end{pmatrix}$$

using $\theta = 45°$

$$\begin{pmatrix} \varepsilon_1 & E_{12} \\ E_{12} & \varepsilon_3 \end{pmatrix} \underbrace{\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}}_{\text{or } \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}} = \begin{pmatrix} \varepsilon_1 \cos\theta + E_{12}\sin\theta \\ E_{12}\sin\theta + \varepsilon_3 \sin\theta \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \varepsilon_1 + E_{12} \\ E_{12} + \varepsilon_3 \end{pmatrix}$$

Component of strain in this direction is:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} \varepsilon_1 + E_{12} \\ E_{12} + \varepsilon_3 \end{pmatrix} = \frac{1}{2}\left[\varepsilon_1 + 2E_{12} + \varepsilon_3\right] = \varepsilon_2$$

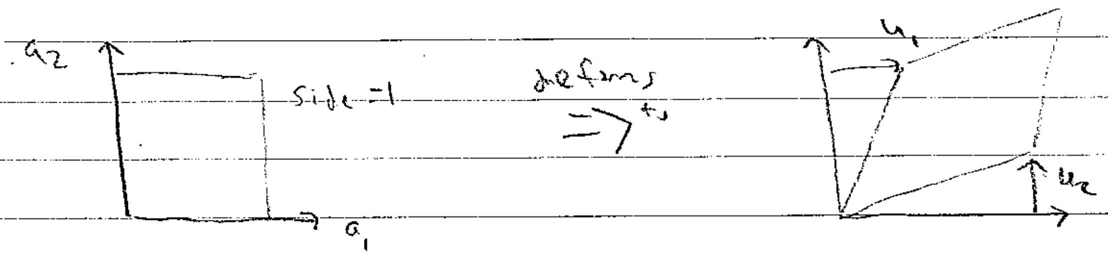
So $\varepsilon_1 + 2E_{12} + \varepsilon_3 = 2\varepsilon_2$

$$2E_{12} = 2\varepsilon_2 - \varepsilon_1 - \varepsilon_3$$

$$E_{12} = \varepsilon_2 - \frac{1}{2}(\varepsilon_1 + \varepsilon_3)$$

$$\boxed{E = \begin{pmatrix} \varepsilon_1 & \varepsilon_2 - \frac{1}{2}(\varepsilon_1 + \varepsilon_3) \\ \varepsilon_2 - \frac{1}{2}(\varepsilon_1 + \varepsilon_3) & \varepsilon_3 \end{pmatrix}}$$

Tutorial Problem 7.



No deformation or change in $\hat{a}_3$ direction: plane strain
So it u_1 and u_2 are proportional with different proportionality constant K

$$u_1 = K_1 a_2 \qquad u_2 = K_2 a_1$$

Find Strain + Rotation tensors, the equivalent rotation vector, the principal strains, the principal axes + the dilatation

Example w/ $K_1 = K_2$ is in lectures

$$E_{3i} = W_{3i} = 0 \text{ for all } i$$

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i}\right)$$

$$E_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial a_1} + \frac{\partial u_1}{\partial a_1}\right) = 0 \qquad E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial a_2} + \frac{\partial u_2}{\partial a_1}\right) = \frac{K_1 + K_2}{2}$$

$$E_{22} = \frac{1}{2}\left(\frac{\partial u_2}{\partial a_2} + \frac{\partial u_2}{\partial a_2}\right) = 0 \qquad E_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial a_1} + \frac{\partial u_1}{\partial a_2}\right) = E_{12}$$

$$E = \begin{pmatrix} 0 & \frac{K_1+K_2}{2} \\ \frac{K_1+K_2}{2} & 0 \end{pmatrix} \qquad W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$W = \begin{pmatrix} 0 & \frac{K_1-K_2}{2} \\ \frac{K_2-K_1}{2} & 0 \end{pmatrix}$$

$$\text{rotation vector } w = -(W_{23}, W_{31}, W_{12})^T = -\begin{pmatrix} 0 \\ 0 \\ \frac{K_1-K_2}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{K_2-K_1}{2} \end{pmatrix}$$

To get the principal strains + principal axes, need to diagonalize the strain tensor:

To diagonalize the tensor, find its eigenvalues + eigenvectors

$E\alpha = \lambda\alpha$

$$\begin{pmatrix} 0-\lambda & \frac{k_1+k_2}{2} \\ \frac{k_1+k_2}{2} & 0-\lambda \end{pmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$$

$$\det\begin{pmatrix} & \\ & \end{pmatrix} = \lambda^2 - \left(\frac{k_1+k_2}{2}\right)^2 = 0$$

$$\left\{\lambda - \left(\frac{k_1+k_2}{2}\right)\right\}\left\{\lambda + \left(\frac{k_1+k_2}{2}\right)\right\} = 0$$

$$\lambda = \pm \frac{k_1+k_2}{2}$$

If $\lambda_1 = \frac{k_1+k_2}{2}$ then
$$\begin{pmatrix} -\left(\frac{k_1+k_2}{2}\right) & \frac{k_1+k_2}{2} \\ \frac{k_1+k_2}{2} & -\left(\frac{k_1+k_2}{2}\right) \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$$

$$-e_1\left(\frac{k_1+k_2}{2}\right) + e_2\left(\frac{k_1+k_2}{2}\right) = 0$$

$e_1 = e_2$ → same eqn twice

$$\lambda_1 = \frac{k_1+k_2}{2} \quad \hat{e}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

For $\lambda_2 = -\left(\frac{k_1+k_2}{2}\right)$
$$\begin{pmatrix} \frac{k_1+k_2}{2} & \frac{k_1+k_2}{2} \\ \frac{k_1+k_2}{2} & \frac{k_1+k_2}{2} \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$$

$$\lambda_2 = -\left(\frac{k_1+k_2}{2}\right) \quad \hat{e}_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$e_1 = -e_2$

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$ = matrix w/ eigenvectors as columns

AEA^T should $= E$

where we have normalized the eigenvectors

check:

$$AEA^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & \frac{k_1+k_2}{2} \\ \frac{k_1+k_2}{2} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} =$$

eigenvectors = principal Axes

$$\begin{pmatrix} \cdot \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}}\left(\frac{k_1+k_2}{2}\right) & \frac{1}{\sqrt{2}}\left(\frac{k_1+k_2}{2}\right) \\ \frac{1}{\sqrt{2}}\left(\frac{k_1+k_2}{2}\right) & \frac{1}{\sqrt{2}}\left(\frac{k_1+k_2}{2}\right) \end{pmatrix}$$

$$\begin{pmatrix} -\frac{k_1+k_2}{2} & 0 \\ 0 & \frac{k_1+k_2}{2} \end{pmatrix}$$

$Q = 0$ principal strains $\pm\frac{k_1+k_2}{2}$

Dilatation = trace of strain $= 0$