2 March 2014 Euan's Tutorial 1 Questions Tensors

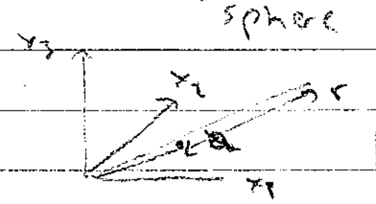
1. Let $\underset{\sim}{a}$ be the position vector of a given point $(x_{10}, x_{20}, x_{30})^T$ and $\underset{\sim}{r}$ be the position vector of any point $(x_1, x_2, x_3)^T$. Describe the locus or $\vec{r}$ if:

a) $|\underset{\sim}{r} - \underset{\sim}{a}| = 3$ a) $\underset{\sim}{r} - \underset{\sim}{a} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} x_{10} \\ x_{20} \\ x_{30} \end{pmatrix} = \begin{pmatrix} x_1 - x_{10} \\ x_2 - x_{20} \\ x_3 - x_{30} \end{pmatrix}$

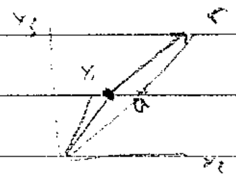
$|\underset{\sim}{r} - \underset{\sim}{a}|$ = length of the vector $= \sqrt{(\underset{\sim}{r} - \underset{\sim}{a}) \cdot (\underset{\sim}{r} - \underset{\sim}{a})^T}$

$= \sqrt{(x_1 - x_{10})^2 + (x_2 - x_{20})^2 + (x_3 - x_{30})^2} = 3$

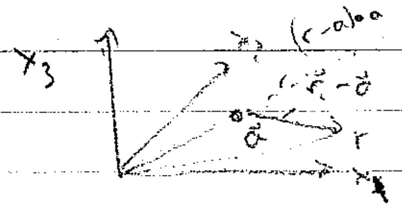
This is a ~~circle~~ sphere of radius 3.



$|\vec{r} - \vec{a}| = 3$ length is always 3



b) $(\vec{r} - \vec{a}) \cdot \vec{a} = 0$:



$\Rightarrow$ if $(\vec{r} - \vec{a} \cdot \vec{a}) = 0$ then $r - a$ must be $\perp$ to $\vec{a}$

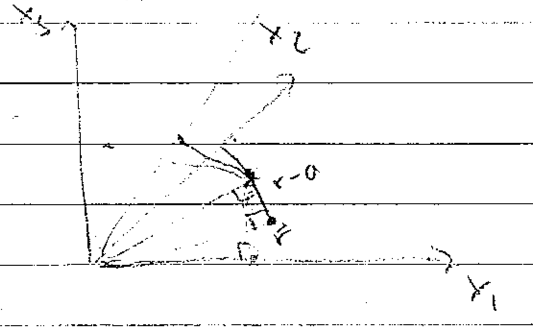
So if we were in 2-D r would be a ~~pt a~~ locus of line $\perp$ to a

But in 3-D r must be any vector in a plane $\perp$ to a

- formally $\vec{r} \cdot \vec{a} - \vec{a} \cdot \vec{a} = 0$ $r\cos\theta = \frac{|a|}{a}$

$r a \cos\theta = |a|$

c) $(\vec{r} - \vec{a}) \cdot \vec{r} = 0$



Now $(\vec{r} - \vec{a})$ must be $\perp$ to $\vec{r}$

Given $\vec{a}$ and

$\vec{r} \cdot \vec{r} = \vec{a} \cdot \vec{r} = 0$

Consider only magnitude! $|r|^2 = |r||a|\cos\theta$

$\frac{|r|^2}{|r|} = a\cos\theta$

$|r| = |a|\cos\theta$

set up another coordinate system so that $\vec{a}$ is along an axis - e.g. $\vec{x}_1$:

Then θ is measured from x_1

Tutorial question 1c: $(\underline{\mathbf{r}}-\underline{\mathbf{a}})\cdot\underline{\mathbf{r}}=0$

- By inspection, two solutions are $\underline{\mathbf{r}}=\underline{\mathbf{a}}$ and $\underline{\mathbf{r}}=0$.
- Set up a coordinate system with $\underline{\mathbf{x}}_{\underline{\mathbf{1}}}$ along $\underline{\mathbf{a}}$
- If we consider the angle between r and a as θ, then both are at $\theta=0$.
- Choosing many values of θ and drawing, points look like sphere of radius a/2 about point (a/2, 0, 0).
- But how to prove? (to go over in tutorial)

Using equations, with $\underline{\mathbf{a}}=a\underline{\mathbf{u}}_1$

- $(\underline{\mathbf{r}}-\underline{\mathbf{a}})\cdot\underline{\mathbf{r}}=(\underline{\mathbf{r}}\cdot\underline{\mathbf{r}})-(\underline{\mathbf{a}}\cdot\underline{\mathbf{r}})=0$
- $\underline{\mathbf{r}}\cdot\underline{\mathbf{r}}=x_1^2+x_2^2+x_3^2$
- $\underline{\mathbf{a}}\cdot\underline{\mathbf{r}}=ax_1$
- $(\underline{\mathbf{r}}\cdot\underline{\mathbf{r}})-(\underline{\mathbf{a}}\cdot\underline{\mathbf{r}})=x_1^2+x_2^2+x_3^2-ax_1$
- But $(x_1-\frac{a}{2})^2=x_1^2+\frac{a^2}{4}-x_1a$
- so $x_1^2-ax_1=(x_1-\frac{a}{2})^2-\frac{a^2}{4}$
- So $(\underline{\mathbf{r}}\cdot\underline{\mathbf{r}})-(\underline{\mathbf{a}}\cdot\underline{\mathbf{r}})=(x_1-a/2)^2+x_2^2+x_3^2-a^2/4=0$
- Which is the equation for a sphere of radius a/2 at the point (a/2,0,0) QED

Tutorial 1 Q 1 end + 2

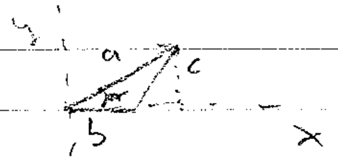
So if $\vec{r}$ is past 90°, it can't go out far enough to be ⊥

- But we are in 3-D, not 2-D so locus is a hemisphere not a semicircle.

2. a) Show that the area of a triangle formed by two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is $\frac{1}{2}\, \underset{\sim}{a} \times \underset{\sim}{b}$

Answer: See notes: p. 22-23 for part b.

For part a)



$\underset{\sim}{a} \times \underset{\sim}{b} = (ab \sin\alpha)\, \hat{n}$ where $\hat{n}$ is ⊥ to a + b

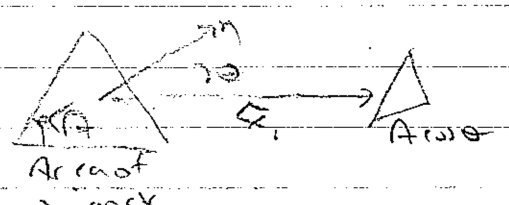
Area of a triangle: $\frac{1}{2}$ base × height.

let angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be α

then the base $= |b|$ and height $= |a| \sin\alpha$

So area $= \frac{1}{2}|a|\,|b| \sin\alpha = \frac{1}{2}|\underset{\sim}{a} \times \underset{\sim}{b}|$ QED

b) So the area of a projected triangle projected along a direction with angle θ to the normal to the triangle:



Show new area = $A \cos\theta$

$A = \frac{|ab \sin\alpha|}{2} = \frac{1}{2}|\underset{\sim}{a}\times\underset{\sim}{b}|$ $\hat{n} = \frac{\underset{\sim}{a} \times \underset{\sim}{b}}{|a \times b|}$ = unit normal due to definition of x-product.

Let the projection direction be in $\hat{x}_1$ direction

(might need to rewrite a + b in new coordinates)

$\underset{\sim}{a} = a_1 \hat{x}_1 + a_2 \hat{x}_2 + a_3 \hat{x}_3 = a_j \hat{x}_j$

projection of a = a_p along $\hat{x}_1$ is

$$a_p = \vec{a} - (\hat{x}_1 \cdot \vec{a})\, \hat{x}_1 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} - \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ a_2 \\ a_3 \end{pmatrix}$$

$$a_p = a_2 \hat{x}_2 + a_3 \hat{x}_3$$

Tutorial One q. 26 continued

Projection for $\underset{\sim}{b}$ along $\hat{x}_1$ is similarly

$$\underset{\sim}{b}_p = b_2 \hat{x}_2 + b_3 \hat{x}_3 \qquad a_p \times b_p = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ 0 & a_2 & a_3 \\ 0 & b_2 & b_3 \end{vmatrix}$$

$\therefore$ area of projected triangle is $A_p = \frac{1}{2} \left| \underset{\sim}{a}_p \times \underset{\sim}{b}_p \right| = \frac{1}{2} \left| (a_2 b_3 - b_2 a_3) \hat{x}_1 + 0 \right|$

$\cos\theta = (\text{projection } \cancel{\text{...}} \text{ direction}) \cdot \hat{n}$

$$\cos\theta = \hat{x}_1 \cdot \hat{n} = (1,0,0) \cdot \frac{\underset{\sim}{a} \times \underset{\sim}{b}}{|\underset{\sim}{a} \times \underset{\sim}{b}|} = \frac{a_2 b_3 - b_2 a_3}{|\underset{\sim}{a} \times \underset{\sim}{b}|}$$

$\cancel{(1,0,0) \cdot (\underset{\sim}{a} \times \underset{\sim}{b})}$

$= x_1$ comp. of $\underset{\sim}{a} \times \underset{\sim}{b}$

$= (a_2 b_3 - b_2 a_3)$

$\therefore a_2 b_3 - b_2 a_3 = |\underset{\sim}{a} \times \underset{\sim}{b}| \cos\theta$

So $A_p = \frac{1}{2} |\underset{\sim}{a} \times \underset{\sim}{b}| \cos\theta$ QED

$= \cos\theta$ (area of original triangle)

Tutorial 1 Question 3 and 4.

3. If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are distinct vectors, construct a RH Cartesian set of axes where x_1 is normal to the plane of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and x_2 and x_3 are any two vectors in the plane of $\underset{\sim}{a}$ and $\underset{\sim}{b}$

To find a vector normal to both $\underset{\sim}{a}$ & $\underset{\sim}{b}$ → which will be in the plane ⊥ to that vector, just use

$$\hat{x}_1 = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \text{unit vector normal to } \underset{\sim}{a} \text{ and } \underset{\sim}{b}$$

let $\hat{x}_2$ be $\frac{\vec{a}}{|\vec{a}|}$ = unit vector along $\underset{\sim}{a}$ direction

then $$\hat{x}_3 = \frac{\hat{x}_1 \times \hat{x}_2}{|x_1 \times x_2|} = \frac{(\vec{a} \times \vec{b}) \times \vec{a}}{|\vec{a} \times \vec{b}|\,|a|}$$

4. Attempt to find a non-trivial solution to $Ax = 0$:

i) for $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 2 \\ -1 & -5 & -3 \end{bmatrix}$

only the trivial solution exists if $\det(A) \neq 0$

Here $\det A = \begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 2 \\ -1 & -5 & -3 \end{vmatrix} = 9 - 8 - 50 - 5 + 24 + 30$
$= 1 - 55 + 54$
$= 0$

$\det A = 0$ so non-trivial solutions exist.

Use Gaussian Elimination to find the simplest ones:

Eqns (a)b(c) with:

$$\left.\begin{matrix} -1 & -5 & -3 & 0 \\ 2 & -1 & 2 & 0 \\ 3 & 4 & 5 & 0 \end{matrix}\right\} \text{rotate rows.}$$

tutorial 1 problem 4 (i) continued

Add multiple of row 1 to other rows to eliminate variables:

$$\begin{matrix} -1 & -5 & -3 & 0 \\ 0 & -11 & -4 & 0 \\ 3 & 4 & 5 & 0 \end{matrix}$$

multiply first row by 2 & add.

multiply 1st row by 3 and add

$$\Downarrow$$

$$\begin{matrix} -1 & -5 & -3 & 0 \\ 0 & -11 & -4 & 0 \\ 0 & -11 & -4 & 0 \end{matrix}$$

last two equations are identical. So all we know is that

$$-11x_2 = 4x_3$$

$$x_2 = -\frac{4}{11}x_3$$

So if $x_3 = 1$ then $x_2 = -\frac{4}{11}x_3$ and $x_1 = -5x_2 - 3x_3$

$$x_1 = -\frac{20}{11}x_3 - 3x_3 = x_3\left(-\frac{53}{11}\right)$$

(ii) $A = \begin{bmatrix} 3 & 4 & 0 \\ 2 & -1 & 0 \\ 0 & 11 & 1 \end{bmatrix}$

$$\det A = \begin{vmatrix} 3 & 4 & 0 \\ 2 & -1 & 0 \\ 0 & 11 & 1 \end{vmatrix} = -3 - 8 = -11 \neq 0$$ So only the trivial solution exists

tutorial 1

Q.5. $\hat{u}_1$ and $\vec{u}_2$ are mutually $\perp$ vectors, used to construct a new coordinate system. $\vec{y}$ is described in the old system.

New system $\vec{y}' = A^T\vec{y}$ where A^T is made of column vectors

Given by the new coordinate system vectors u_1, u_2, u_3 expressed in the old system

$$\vec{u}_1 = \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \end{pmatrix} \quad \vec{u}_2 = \begin{pmatrix} u_{21} \\ u_{22} \\ u_{23} \end{pmatrix} \quad \hat{u}_3 = \begin{pmatrix} u_{31} \\ u_{32} \\ u_{33} \end{pmatrix}$$

where $\vec{u}_3 = \dfrac{\vec{u}_1 \times \vec{u}_2}{|u_1 \times u_2|}$

$$\text{So } A^T = \begin{pmatrix} u_{11} & u_{21} & u_{31} \\ u_{12} & u_{22} & u_{32} \\ u_{13} & u_{23} & u_{33} \end{pmatrix}^T = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}$$

$\uparrow A^T$

$$y' = \begin{pmatrix} \hat{u}_1 \cdot \vec{y} \\ \hat{u}_2 \cdot \vec{y} \\ \hat{u}_3 \cdot y \end{pmatrix} = \begin{matrix} u_{11}y_1 + u_{12}y_2 + u_{13}y_3 \\ u_{21}y_1 + u_{22}y_2 + u_{23}y_3 \\ u_{31}y_1 + u_{32}y_2 + u_{33}y_3 \end{matrix}$$

$$y' = A^T y = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

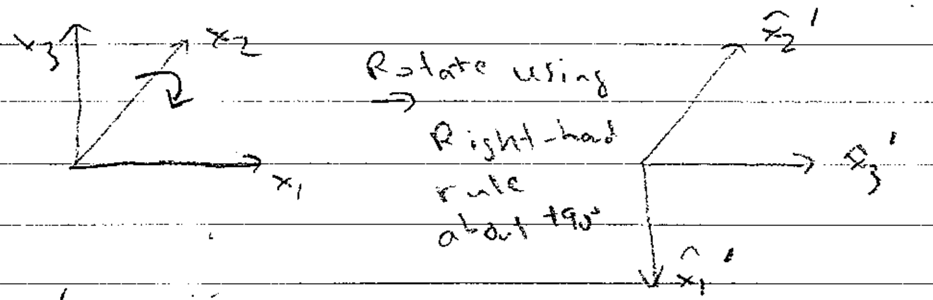
Usually finish Tutorial 1 here.

Tutorial 1 (possibly discussed in Tutorial 2)

6. Construct transformation matrices A for giving the coordinates of a vector $\underline{r}$ in a new coordinate system using the convention

$$\underline{r}(\text{new}) = A^T \underline{r}(\text{old})$$

a) Rotation through $90°$ about $\hat{x}_2$ axis:



$$x_1' \Rightarrow = -x_3$$
$$x_2' = x_2$$
$$x_3' = x_1$$

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

multiply A^T by $\hat{x}_2 \Rightarrow x_2$ $\quad A^T$ by $x_1 \Rightarrow$

$$\hat{x}_1' = A^T \hat{x}_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

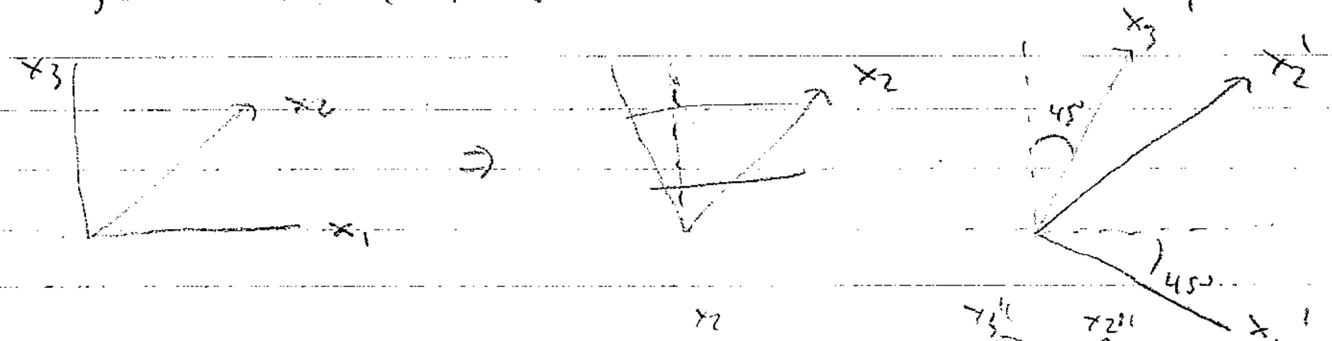
$\Rightarrow$ So the coordinate that was previously called x_1 is now x_3'

$$x_3' = A^T \hat{x}_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

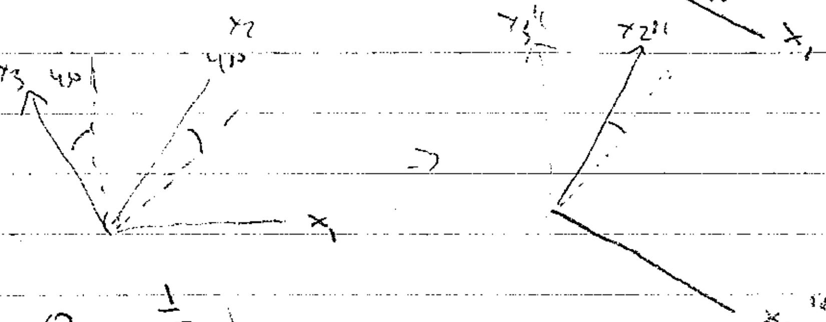
= coordinate axis that was previously $\hat{x}_3$ is now called $-\hat{x}_1$

Tutorial One

6b) Rotation through 45° about $\hat{x}_2$ followed by rotation through 45° about (new) $\hat{x}_1$



$\Rightarrow$ old $x_1, x_2, x_3 \rightarrow$ (intended)



First time: $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}$

$\sin 45° = \cos 45°$
$= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

Next time $A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ $A'^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

A total $= A'^T A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$

$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

If old $x_2 \rightarrow$

Exercises

1) Show that $A\vec{x} = \lambda I\vec{x}$ has a non-trivial solution iff $|A-\lambda I| = 0$

A is an $n \times n$ (square) matrix.

There is a theorem that any set of equations ~~has a non-trivial~~ ~~solution~~: $A\vec{x} = 0$

Has a non-trivial solution only if $\det A = 0$

So $Ax - \lambda x = (A - \lambda I)\vec{x} = 0$ iff $\det(A - \lambda I) = 0$

Suppose $\det A \neq 0$ then

⇒ If A is an $n \times n$ matrix then the following statements are equivalent:

a) A is invertible

b) $Ax = 0$ has only the trivial solution

c) $AX = B$ is consistent for every $n \times 1$ matrix B

2) Eigenvalues λ and corresponding eigenvectors $\vec{x}$ for the matrix

$$A = \begin{bmatrix} 2 & 0 & 6 \\ 0 & 1 & 0 \\ 6 & 0 & -4 \end{bmatrix} \qquad (A - \lambda I)\vec{x} = \begin{bmatrix} 2-\lambda & 0 & 6 \\ 0 & 1-\lambda & 0 \\ 6 & 0 & -4-\lambda \end{bmatrix}$$

det (A) = each element of 1 row multiplied by adjoint

$$\text{so } |A - \lambda I| = -0 \begin{vmatrix} 0 & 6 \\ 0 & -4-\lambda \end{vmatrix} + (1-\lambda)\begin{vmatrix} 2-\lambda & 6 \\ 6 & -4-\lambda \end{vmatrix} + 0 \begin{vmatrix} 2-\lambda & 0 \\ 6 & 0 \end{vmatrix}$$

$$= (1-\lambda)\left[(2-\lambda)(-4-\lambda) - 36\right]$$

$$= (1-\lambda)\left\{-8 - 2\lambda + 4\lambda + \lambda^2 - 36\right\}$$

$$= (1-\lambda)\left\{\lambda^2 + 2\lambda - 44\right\} = 0$$

eigenvalues are $\lambda = 1$ or $\lambda = \frac{-2 \pm \sqrt{4 + 4(44)}}{2} = -1 \pm \sqrt{45}$

if $\lambda = 1$, eigen vector is found by

$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 0 & 0 \\ 6 & 0 & -5 \end{bmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

$$\Rightarrow \quad e_1 + 6e_3 = 0$$
$$6e_1 - 5e_3 = 0 \quad \Rightarrow$$
$$\Downarrow$$
$$0 + -36e_3 - 5e_3 = 0$$
$$-41e_3 = 0$$
$$e_3 = 0 \qquad e_2 = \text{anything so } e_2 =$$
$$e_1 = 0$$

eigen vector $= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

for $\lambda = -1 + \sqrt{45}$

$$\left(A - (-1 + \sqrt{45})I\right) \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = 0$$

$$\begin{matrix} 2 + 1 - \sqrt{45} & 0 & 6 \\ 0 & 1 - (-1 + \sqrt{4}) & 0 \\ 6 & 0 & -4 - 1 - \sqrt{4} \end{matrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \qquad e_2 = 0$$

$$\left(\begin{matrix} (3 - \sqrt{45})e_1 + 6e_3 = 0 \\ 6e_1 - 5 + \sqrt{45}e_3 = 0 \end{matrix} \right) \Rightarrow \begin{bmatrix} a_{11} \\ 0 \\ a_{12} \end{bmatrix}$$

for $\lambda = -1 - \sqrt{45}$.

$$\left(\quad \right)$$